CONCERNING THE BOUNDARIES OF DOMAINS OF A CONTINUOUS CURVE*

BY W. L. AYRES

We shall consider a space M consisting of all the points of a plane continuous curve M†, and all point sets mentioned are assumed to be subsets of M. A connected set of points D of M is said to be an *M-domain* if $M-D$ is closed. The set of all limit points of D which do not belong to D is called the *M-boundary* of D. If D is an M-domain, D' denotes the M-boundary of D, and $\overline{D}$ denotes the set $D+D'$. The M-boundary of an M-domain D is closed but not necessarily connected, even if D is simply connected, as we may easily show by examples. If N is a continuum, a maximal connected subset of $M-N$ is called a *complementary M-domain* of N.

THEOREM I.‡ *Every closed and connected subset of the M-boundary of a complementary M-domain of a continuous curve N is a continuous curve.*

PROOF. Let K denote a closed and connected subset of the M-boundary of a complementary M-domain D of N. Suppose K is not connected im kleinen. Then there exist§ two concentric circles C_1 and C_2 (let r_i denote the radius of C_i and let $r=r_1-r_2>0$) and an infinite sequence of subcon-

* Presented to the Society, February 26, 1927.

† A point set which is closed, connected and connected im kleinen is called a *continuous curve*. In general it may be either bounded or unbounded.

‡ For the case where M is the entire plane and N is bounded, see R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 361.

§ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296. This theorem is stated by Moore for a bounded continuum, but the theorem remains true without the condition of boundedness.

tinua of K, K_∞, K_1, K_2, K_3, $\cdots$, such that (1) each of these continua $K_\alpha(\alpha=\infty, 1, 2, \cdots)$ contains a point a_α on C_1 and a point b_α on C_2, but no point exterior to C_1 or interior to C_2, (2) no two of these continua have a point in common and no one of them, except possibly K_∞, is a proper subset of any connected subset of K which contains no point without C_1 or within C_2, (3) the set K_∞ is the sequential limiting set of the sequence of continua K_1, K_2, K_3, $\cdots$. For any i, let $K_i^*=K_\infty+K_{i+1}+K_{i+2}+\cdots$. Let d_1 be the smaller of the numbers $\frac{1}{4}r$ and $\frac{1}{2}d(K_1, K_1^*)$†.

Let R_1 denote the set of all points $[P]$ of N such that the distance from P to some point of K_1 is less than d_1. The set R_1 is an open subset of N and hence R_1 contains an arc from a_1 to b_1.‡ This arc contains a subarc x_1y_1 such that x_1 is on C_2 and y_1 is on C_1 and every other point of x_1y_1 is between C_1 and C_2.

For each $n>1$, let d_n be the smallest of the numbers d_{n-1}, $r2^{-n-1}$, and $\frac{1}{2}d(K_n, K_n^*+x_1y_1+x_2y_2+\cdots+x_{n-1}y_{n-1})$. Let R_n denote the set of all points $[P]$ of N such that there is some point of K_n whose distance from P is less than d_n. The set R_n contains an arc from a_n to b_n and this arc contains a subarc x_ny_n such that x_n is on C_2, y_n is on C_1, and every other point of x_ny_n is between C_1 and C_2.

There exists an increasing sequence of positive integers n_1, n_2, n_3, $\cdots$, such that (1) C_2 contains three points X_1, X_2, X_3 such that every point x_{n_i} lies on the arc $X_1X_2X_3$ of C_2 and in the order $X_1X_2x_{n_1}x_{n_2}\cdots X_3$, (2) C_1 contains three points Y_1, Y_2, Y_3 such that every point y_{n_i} lies on the arc $Y_1Y_2Y_3$ of C_1 and in the order $Y_1Y_2y_{n_1}y_{n_2}\cdots Y_3$, (3) X_3 is the sequential limit point of $[x_{n_i}]$ and Y_3 is the sequential limit point of $[y_{n_i}]$. Clearly K_∞ contains X_3 and Y_3.

---

† If A and B are sets of points, the symbol $d(A,B)$ denotes the lower bound of all numbers $d(x,y)$, where x is a point of A, y is a point of B, and $d(x,y)$ is the distance from x to y.

‡ R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 255.

The set K_∞ contains points X and Y on the circles which are concentric with C_1 and with radii $r_2+r/10$ and $r_1-r/10$ respectively. Let η be the smaller of the two numbers $r/10$ and r_2. Since N is connected im kleinen, there exists a positive number δ_η such that any point of N within a distance δ_η of X or Y may be joined to X or Y, as the case may be, by an arc of N every point of which is within a distance η of X or Y. Let n_s be the smallest integer such that $x_{n_s}y_{n_s}$ contains two points P and Q such that $d(P, X)<\delta_\eta$ and $d(Q, Y)<\delta_\eta$. Then N contains arcs PX and QY, every point of which is within a distance η of X and Y respectively. Let order be defined on these arcs as *from P to X* and *from Q to Y*. The arcs PX and QY have points in common with every arc $x_{n_i}y_{n_i}$ for $i\geqq s$.

Let V_{10} and V_{20} be the last points the arcs PX and QY have in common with $x_{n_s}y_{n_s}$. Let U_{11} and U_{21} be the first points and V_{11} and V_{21} be the last points the subarcs $V_{10}X$ and $V_{20}Y$ have in common with $x_{n_{s+1}}y_{n_{s+1}}$. Let U_{12} and U_{22} be the first points the subarcs $V_{11}X$ and $V_{21}Y$ have in common with $x_{n_{s+2}}y_{n_{s+2}}$. The set J, composed of the arcs $V_{10}V_{20}$ of $x_{n_s}y_{n_s}$, $U_{11}V_{11}$ and $U_{21}V_{21}$ of $x_{n_{s+1}}y_{n_{s+1}}$, $U_{12}U_{22}$ of $x_{n_{s+2}}y_{n_{s+2}}$, $V_{10}U_{11}$ and $V_{11}U_{12}$ of PX, $V_{20}U_{21}$ and $V_{21}U_{22}$ of QY, is a simple closed curve. The subarc of $x_{n_{s+1}}y_{n_{s+1}}$ lying within J and the arc $x_{n_{s+3}}y_{n_{s+3}}$ have points p_1 and p_3, respectively, in common with the circle concentric with C_1 and with radius $r_2+\frac{1}{2}r$. The point p_1 is interior to J and within a distance $d_{n_{s+1}}$ of some point of K and this point is a limit point of D. Thus D contains a point in the interior of J. Similarly D contains a point within a distance $d_{n_{s+3}}$ of p_3 and thus in the exterior of J. Since D is connected D must contain a point of J. But as J belongs to N and D to $M-N$, D cannot contain a point of J. Thus the assumption that K is not connected im kleinen has led to a contradiction.

THEOREM II. *If a maximal connected subset K of the boundary of an M-domain is a continuous curve, every closed and connected subset of K is a continuous curve.*

DEFINITION. If D is an M-domain and P is a point of $M-\overline{D}$, the M-boundary of the maximal connected subset of $M-\overline{D}$ containing P will be called the *M-boundary of D with respect to P*. This is a generalization of the notion of outer boundary as defined by R. L. Moore.* If M is the entire plane, D is bounded, and P is a point of the maximal connected subset of $M-\overline{D}$ which is unbounded, then the M-boundary of D with respect to P is exactly the outer boundary of D as defined by Moore.*

THEOREM III.† *If D is an M-domain, P is a point of $M-\overline{D}$, and B is the M-boundary of D with respect to P, then B is the entire M-boundary of some M-domain which contains D.*

PROOF. The entire set D lies in the same maximal connected subset of $M-B$ and let R denote this maximal connected subset. Evidently R is an M-domain containing D. Suppose Q is a point of the M-boundary of R. If Q does not belong to B, Q belongs to a maximal connected subset of $M-B$ which is different from R. Then Q is not a limit point of R.‡ Therefore every point of the M-boundary of R is a point of B. Conversely every point of B is an M-boundary point of D and thus of R. Hence B is identical with the M-boundary of R, an M-domain containing D.

THEOREM IV. *If (1) D is an M-domain and P is a point of $M-\overline{D}$, (2) every maximal connected subset of D' is a continuous curve, (3) the M-boundary of D with respect to P, which we denote by B, is bounded, then every maximal connected subset of B is either a point, a simple continuous arc or a simple closed curve.*

PROOF. Let R be the maximal connected subset of $M-\overline{D}$ containing P, and let B_1 be a maximal connected subset of B.

---

* *Concerning continuous curves in the plane*, loc. cit., p. 256.

† Compare R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 3, p. 258.

‡ See R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), Lemma 1, p. 302.

By Theorem II, B_1 is a continuous curve and B_1 is bounded by hypothesis. If B_1 consists of a single point, our theorem is proved. If B_1 consists of more than a single point then by a theorem due to Mazurkiewicz,* B_1 contains two points x and y which do not cut B_1. The continuous curve B_1 contains an arc xzy from x to y. If $B_1 \equiv xzy$, our theorem is proved. If not, let p be a point of B_1 which does not lie on xzy. By Theorem III, B is the entire M-boundary of some M-domain H which contains D. Clearly R and H are mutually exclusive and B is the entire M-boundary of each. By a theorem due to Wilder,† if p_1 and p_2 are points of R and H respectively there exist arcs p_1x and p_1y which lie except for x and y wholly in R and arcs p_2x and p_2y which lie except for x and y wholly in H. The sets p_1x+p_1y and p_2x+p_2y contain arcs xuy and xvy which lie wholly in R and H respectively except for the points x and y.

Let J_1, J_2, J_3 be the simple closed curves formed of $xuy+xzy$, $xvy+xzy$, $xuy+xvy$ respectively and let I_i and E_i denote the interior and exteiior of $J_i (i=1, 2, 3)$. We have three cases to consider:

Case (1). Suppose $I_3 = I_1 + I_2 + <xzy>$.‡ Any point q of $B - xzy$ lies either in I_1, I_2 or E_3. If q lies in I_1, I_1 contains a point of H since q is a limit point of H. The exterior E_1 contains $<xvy>$ of H. But H is connected and contains no point of J_1. Hence I_1 contains no point of $B - xzy$. Similarly I_2 contains no point of $B - xzy$. Then every point of $B - xzy$ lies in E_3. Since x and y are not cut-points of B_1, the continuous curve B_1 contains an arc px which does not contain y and an arc py which does not contain x. The set $px+py$ contains an arc xwy from x to y. Since p is in E_3 and no point of J_3 except x and y is a point of B_1, the set $<xwy>$ lies

* *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae vol. 2 (1921), pp. 119–130.

† Loc. cit., Theorem 1, p. 342.

‡ If xzy denotes a simple continuous arc with end-points x and y $<xzy>$ denotes $xzy-(x+y)$.

entirely in E_3, and thus the arcs xwy and xzy have only x and y in common. Let J_4 be the simple closed curve $xzy+xwy$. We will show that $B-J_4$ is vacuous and thus prove $B \equiv B_1 \equiv J_4$. Suppose $B-J_4$ contains a point q_1. If q_1 lies in the interior of J_4 both R and H have points in the interior of J_4 since q_1 is a limit point of both domains. One of the two sets $<xuy>$ or $<xvy>$, say $<xuy>$, lies entirely in the exterior of J_4. Then R contains points interior and exterior to J_4 but contains no point of J_4, which is impossible. If q_1 lies in the exterior of J_4, both R and H contain points in the exterior of J_4, and one of them contains points in the interior, which is impossible. Therefore $B-J_4$ is vacuous, which proves the theorem for this case.

Case (2). Suppose $I_2 = I_1 + I_3 + <xuy>$.

Case (3). Suppose $I_1 = I_2 + I_3 + <xvy>$.

In Cases (2) and (3), it may be proved by methods similar to those of Case (1) that $B \equiv B_1$ and B_1 is a simple closed curve. Therefore B_1 is either a point, a simple continuous arc, or a simple closed curve.

In proving Theorem IV we have obtained this result:

THEOREM V. *Under the hypothesis of Theorem* IV, *if any maximal connected subset J of the M-boundary of D with respect to P is a simple closed curve, then J is the entire M-boundary of D with respect to P.*

THEOREM VI. *If D is an M-domain, P is a point of $M-\overline{D}$, R is the maximal connected subset of $M-\overline{D}$ containing P, and Q is a point of the maximal connected subset of $M-\overline{R}$ which contains D, then R' is the M-boundary of R with respect to Q.*

PROOF. Let H denote the maximal connected subset of $M-\overline{R}$ which contans D. By definition H' is the M-boundary of R with respect to Q. The set H' is a subset of R' since the M-boundary of a domain with respect to a point is always a subset of the M-boundary of the domain. By definition, R' is the M-boundary of D with respect to P. Then every point of R' is a limit point of D and thus of H. As H is a subset of

$M-\overline{R}$, no point of R' is a point of H. Therefore every point of R' is an M-boundary point of H, that is, R' is a subset of H'. Hence $R' \equiv H'$.

THE UNIVERSITY OF PENNSYLVANIA

A THEOREM ON CONNECTED POINT SETS

BY C. KURATOWSKI AND C. ZARANKIEWICZ

1. *Introduction.* The purpose of this paper is to prove the following theorem.

THEOREM. *If S is a connected point set and Z is the set of all points such that $S-p$ is neither connected nor the sum of two connected sets, then Z is finite or countable.*

2. LEMMA. *If S, P, and Q are three non-vacuous connected sets (or points), and if*

(1) $P + Q \subset S$, (2) $P \cdot Q = 0$,

(3) $A \subset S - P$, (4) $B \subset S - Q$,

(5) $A \cdot Q = 0$, (6) $B \cdot P = 0$,

(7) *A and $S-P-A$ are mutually separated,*

(8) *B and $S-Q-B$ are mutually separated,*

then $A \cdot B = 0$.

PROOF. By (1) and (3), $A+P \subset S$. Hence, by (2) and (5), $A+P=(A+P)\cdot(S-Q)$. By (4), $S-Q=B+(S-Q-B)$. Therefore

$$A + P = (A + P)\cdot B + (A + P)\cdot(S - Q - B). \tag{9}$$

It follows from (2) and (6) that $P=P-Q-B \subset (A+P)$ $(S-Q-B)$. Since $P\neq 0$, we have

$$(A + P)\cdot(S - Q - B) \neq 0. \tag{10}$$

Now, by (8), the sets $(A+P)\cdot B$ and $(A+P)\cdot(S-Q-B)$ are mutually separated. On the other hand, by virtue of a