THE CRITICAL POINTS OF FUNCTIONS AND THE CALCULUS OF VARIATIONS IN THE LARGE*

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1. *Introduction.* The conventional object in a paper on the calculus of variations is the investigation of the conditions under which a maximum or minimum† of a given integral occurs. Writers have accordingly done little with extremal segments that have contained more than one point conjugate to a given point. An extended theory is needed for several reasons.

One reason is that in applying the calculus of variations to that very general class of dynamical systems or differential equations which may be put in the form of the Euler equations, it is by no means a minimum or a maximum that is always sought. For example, if in the problem of two bodies we make use of the corresponding Jacobi principle of least action the ellipses which thereby appear as extremals always have pairs of conjugate points on them, and do not accordingly give a minimum to the integral relative to neighboring closed curves, so that no example of periodic motion would be found by a search for a minimum of the Jacobi integral. In general if one is looking for extremals joining two points, or periodic extremals deformable into a given closed curve, the a priori expectation, as justified by the results of this paper, in general problems, would seem to be that many more solutions would fail to give a minimum than would give a minimum. Even if the ultimate object is

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† For work on the absolute minimum see Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437, and Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2. Further references will be found in these works.

narrowed down to the search for minima the existence of such minima is tied up in the large with the existence of extremals which do not furnish minima.

However, the real feeling of the author is that the classical calculus of variations is both logically and esthetically incomplete. The object of the present paper is to show by selected theorems and examples something of the more extended results now within reach of the science.* The author wishes to say, however, that in view of the tremendous technical difficulties involved, and the scope of the investigations now being carried on it is bound to be several years before a final idea of the possibilities of the new theory can be realized.

No theory in the large can well escape analysis situs. It is not to be expected, however, that pure analysis situs should contain all the results necessary for all the applications. For the present purposes a fundamental paper in analysis and analysis situs is the author's paper cited below.†

To reduce the study of the existence and nature of extremals to the study of critical points, admissible curves in any problem in the calculus of variations are replaced by nearby curves composed of a succession of short extremals.‡ The value of the given integral along the resulting broken extremal g is regarded as a function f of the coordinates of the end points of the component short extremals. Critical points of f occur when and only when g has no corners.

* Of the earlier work the most significant from the points of view of this paper is the "minimax principle." See Birkhoff, Colloquium Publications of this Society, vol. 9, *Dynamical Systems*, pp. 133–139.

† Marston Morse, *Relations between the critical points of a real function of n independent variables*, Transactions of this Society, vol. 27 (1925), pp. 345–396. This paper will be referred to as Morse I.

A *critical point* of a function is a point at which all of its first partial derivatives vanish.

‡ This idea is doubtless rather old. The manner in which it has been applied is variable. See Birkhoff, loc. cit. In proving his own theorems in his work not yet published the author has had occasion to go into many questions connected with the application of this idea that have never previously been considered.

Thus the search for extremals is reduced to the search for critical points of f.

Up to the present, a study has been made of three fields, first of extremals joining two fixed points, second of closed extremals, and third of extremals from a point to a manifold cut transversally by the manifold.

The first of these fields occupies the principal place in the classical calculus of variations. The second is of central importance for the theory of dynamical systems. The third leads to a geometry suggestive of real algebraic geometry without the algebra. Particularly interesting are the relations in the large that may be found between straight lines, hyperplanes, hyperspheres, and closed manifolds.

2. *The Hypotheses on the Integrand.** Let S be a closed region in the space of the variables $(x_1, \cdots, x_m) = (x)$. Let

$$F(x_1, \cdots, x_m, r_1, \cdots, r_m) = F(x,r) \tag{1}$$

be a positive, analytic function of its arguments, for (x) on S, and (r) any set not (0). Suppose further that F is positively homogeneous of order one in the variables (r). We employ the parametric form, taking $F(x, x')$ as our integrand, where (x') stands for the set of derivatives of (x) with respect to the parameter t. We suppose that the problem is ***positively regular***, that is, that

$$\sum_{ij} F_{r_i r_j}(x,r)\eta_i\eta_j > 0, \quad (i,j = 1,2, \cdots, m), \tag{2}$$

* For that which concerns the classical theory of the calculus of variations in space the reader may refer to the following articles:

Mason and Bliss, *The properties of curves in space which minimize a definite integral*, Transactions of this Society, vol. 9 (1908), pp. 440–466.

Bliss, *The Weierstrass E-function for problems of the calculus of variations in space*, Transactions of this Society, vol. 15 (1914), pp. 369–378.

Bliss, *Jacobi's condition for problems of the calculus of variations in parametric form*, Transactions of this Society, vol. 17 (1916), pp. 195–206.

Hadamard, *Leçons sur le Calcul des Variations*, vol. 1, Paris, 1910.

Carathéodory, *Die Methode der geodätischen äquidistanten und das Problem von Lagrange*, Acta Mathematica, vol. 47 (1926), pp. 199–236.

for (x) and (r) as before, and (η) any set not (0) nor proportional to (r).

In general, for the theorems in the small in this paper, the hypothesis $F>0$ may be omitted, and the hypothesis that F be analytic replaced by the requirement that F be of class C'''. Further details will not be given here.

3. *Conjugate Points and their Orders.* Our future work depends not only upon a suitable definition of conjugate points but also of their orders.

Let there be given an extremal g which is an ordinary analytic curve joining a point P to a point Q. Let λ be the direction of g at P. Let the direction cosines of the directions neighboring λ be admissibly* represented as functions of $(m-1)$ parameters (α). Let s be the arc length of the extremal through P measured from P in the direction determined by (α). On the extremals issuing from P with the directions determined by (α) the coordinates (x) will be analytic functions of s and (α). Let $\Delta(s)$ be the jacobian of the coordinates (x) with respect to s and (α), evaluated for the parameters (α) that determine λ.

Each point on g, not P, at which $\Delta(s)$ vanishes, is called a conjugate point of P, and the order of the vanishing of $\Delta(s)$ the order of that conjugate point.

The order of a conjugate point is shown to be invariant† under admissible space transformations, and to be at most $m-1$. A conjugate point of the rth order will be counted as if it were r distinct conjugate points.

4. *The Type Number of an Extremal.* We shall first consider extremals joining two fixed points P and Q. Before turning to a theory of all such extremals it is necessary to consider one such extremal alone. Suppose g is an ordinary

* That is, the direction cosines are to be analytic functions of the parameters (α) of such sort that not all of the jacobians of $m-1$ of the direction cosines with respect to the parameters (α) are zero.

† Although conjugate points have been defined in various ways the notion of the order of a conjugate point does not seem to have received the attention it requires.

analytic curve without multiple points. The author has shown how g and a neighborhood of g can be mapped analytically, on a straight line segment h and a neighborhood of h. For present purposes then we can suppose that g is a segment of the x_1 axis.

Let us cut across g with n hyperplanes perpendicular to g. These hyperplanes divide g into $n+1$ successive segments. Suppose the hyperplanes are placed so near together that no one of these $n+1$ segments contains a conjugate point of its initial end point. Let us number these hyperplanes in the order of increasing x_1. Let P_i be any point near g on the ith one of these hyperplanes. Let the points

$$P,\ P_1,\ P_2,\ \cdots,\ P_n,\ Q$$

be successively joined by extremal segments neighboring g, and let the resulting broken extremal be denoted by E. Let $[u]$ be the set of the coordinates, other than x_1, of the points P_i. The value of the given integral J taken along E will be an analytic function of the variables $[u]$, and will be denoted by $J(u)$.

The function $J(u)$ will have a critical point when $[u]=[0]$. Let H be the quadratic form making up the terms of second order in $J(u)$ in a Taylor's expansion about $[u]=[0]$. We have the following fundamental theorem.*

If Q is a conjugate point of P of the rth order, the rank of H is $p-r$. If Q is not conjugate to P the rank of H is p, and its type number† *is the sum of the orders of the conjugate points of P preceding Q, where $p=(m-1)n$.*

Note particularly that the type number of H will be unchanged if we increase n to any arbitrary integer. The pre-

* For the case $m=2$, the proof of this theorem has already appeared See Marston Morse, *The foundations of a theory in the calculus of variations in the large*, Transactions of this Society, vol. 30 (1928), pp. 213–274. This paper will be referred to as Morse II.

† By the type number of H is meant the number of negative coefficients appearing in H when H has been reduced by a real, non-singular, linear transformation to squared terms only.

ceding theorem was developed primarily for use in the following theory in the large. Incidentally, it may be used to give what is probably the definitive answer as to the generalization of the famous Sturm separation theorem, for the case of systems of m ordinary, second order, self-adjoint, linear, differential equations in m dependent variables.

5. *Conjugate Points characterized by Deformations.** The entities deformed are families of curves joining P to Q and depending on μ parameters. They are called *μ-families.* The μ parameters are supposed represented by a point P on a closed μ-dimensional complex† C_μ. Each point P on C_μ corresponds to a curve of the family. The curves are supposed to vary continuously with the position of P on C_μ.

Let g be the given extremal, and let J_0 be the value of J along g. Suppose Q is not conjugate to P, but that there are k points $(k>0)$ conjugate to P and preceding Q. We are going to consider continuous deformations of μ-families subject to the condition,

$$J \leqq J_0 - e^2, \qquad e > 0, \tag{3}$$

imposed on each member of the μ-family. We have the following characterization of the number of conjugate points in terms of deformations.

Corresponding to any sufficiently small neighborhood N of g, there exists within N an arbitrarily small neighborhood N_1 of g, and an arbitrarily small positive constant e, with the following properties. The type number k is the least integer k for which the $0, 1, \cdots, (k-2)$*-families on N_1, satisfying* (3), *can be deformed into a single curve on N, subject to* (3), *and for which there exists at least one $(k-1)$-family which cannot be so deformed into a single curve.*

* The theory of these deformations in the small has already been developed in the paper Morse II for the case $m=2$. The work appearing there for $m=2$ applies with hardly any change for a general m.

† Terms referring to pure analysis situs are used in the senses defined by Veblen in The Cambridge Colloquium, 1916, Part II, *Analysis Situs.*

A theorem similar to this but without use of the genera notion of μ-families is given by Birkhoff* for the case of closed trajectories of dynamics when $k=1$. For $k=1$ our theorem refers to a 0-family of curves joining P to Q, that is to a pair of curves joining P to Q. The theorem states that there exists a pair of such curves in N_1, which satisfy (3), but which cannot be deformed into each other subject to (3) without passing out of N.

A simple example of the theorem is obtained by considering the integral of the arc length of an n-sphere. Let the extremal g be an arc of a great circle slightly longer than a semi-circle. If $n=2$, then $k=1$, and the statement of the preceding paragraph is clearly true. For a general n, $k=n-1$, and one can apply the theorem as stated.

If $k=0$, the theorem says nothing. However, the theorem of the calculus of variations commonly known as Osgood's Theorem† is seen to fill the gap in a most natural way.

6. *Theorems in the Large.* The preceding theorem is a typical deformation theorem in the small. One can also consider the mutual relations of μ-families in the large. In particular one can introduce the notion of "bounding" or "non-bounding" families. It is by means of such considerations that one arrives at the following theorems in the large.

Suppose the region S homeomorphic with the interior and boundary of an $(m-1)$-sphere, and that S's boundary is *extremal convex*. That is, suppose there exists a positive constant e so small that any extremal segment on which $J<e^2$ and which joins two boundary points will lie interior to S except at most for its end points.

If P_0 and Q_0 are any two points of S there exist in their respective neighborhoods points P and Q which are joined by no extremals on which P is conjugate to Q.

Such a pair of points will be termed *non-specialized.* For such a pair there will be at most a finite set of extremals

* Birkhoff, *Dynamical systems with two degrees of freedom*, Transactions of this Society, vol. 18 (1917), p. 249.

† Osgood, Transactions of this Society, vol. 2 (1901), p. 273.

joining P to Q with lengths less than a given constant, although altogether there may be an infinite number of extremals joining P to Q among the extremals of all lengths. For a non-specialized pair of points the number of conjugate points of P prior to Q will be called the *type number* of g. We have the following theorem.

If on the extremals joining two non-specialized points P and Q, J is bounded, then there will be at most a finite set of extremals joining P to Q, and between the integers M_k which give the total number of extremals of type k joining P to Q, the following relations will hold true:

$$
\begin{aligned}
&1 \leqq M_0,\\
&1 \geqq M_0 - M_1,\\
&1 \leqq M_0 - M_1 + M_2,\\
(4)\quad &1 \geqq M_0 - M_1 + M_2 - M_3,\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad ,\\
&1 \leqq [M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^{r-1} M_{r-1}](-1)^{r-1},\\
&1 = M_0 - M_1 + M_2 - M_3 + \cdots + (-1)^r M_r.
\end{aligned}
$$

Simple examples will show that there may perfectly well be an infinite number of extremals joining two non-specialized points P and Q. Here the following more general theorem holds.

Corresponding to any positive constant a there exists a positive constant b with the following properties. If P and Q are two non-specialized points the relations (4) *will still be true if M_k be taken as the number of extremals of type k for which $J<a$, plus the number of extremals of type k in a suitably chosen subset of the extremals on which $a \leqq J < b$.*

Let P and Q be any pair of points, specialized or not. Let N_k be the number of extremals joining P to Q on which there are k points conjugate to P. If there are an infinity of such extremals we replace N_k by ∞. The sequence $N_0, N_1, N_2, \cdots$ will be called the *conjugate number sequence* for P and Q. If P and Q are non-specialized, we can readily prove the following.

If there are N_k extremals of type $k>1$, there are at least N_k extremals of the two adjacent types.

If there are N_1 extremals of type 1, *there are at least N_1+1 extremals of the two adjacent types.*

If there are N_0 extremals of type 0, *there are at least N_0-1 extremals of the adjacent type* 1.

These statements still hold if any of the N's are ∞, understanding $\infty-1$ or $\infty+1$ as standing merely for an infinity. Many other facts can be proved about the conjugate number sequence. One of the most curious as well as important facts is the following.

If P and Q are non-specialized and if any one of the symbols N_k is zero, then either at least two of the preceding symbols are ∞, or else the set of preceding symbols satisfies (4).

Suppose P and Q are any two points on S, specialized or non-specialized. The extremals joining P to Q may very well be infinite in number. In any case the extremals which joint P to Q, and for which J is less than a constant J_0, will either be finite in number, or else can be grouped into a finite set of r-parameter, connected, and in general analytic families, for which $r<m-1$, and on which J is a constant. Each isolated extremal joining P to Q to which a type number has not already been assigned, and each family of extremals joining P to Q, can be counted as if it were a finite set of extremals of definite types, and so counted all the preceding relations between the integers M_k still hold.

A strong aid in proving the existence of extremals joining any two points P and Q, whether specialized or not, is the following principle.

If in the respective neighborhoods of two points P_0 and Q_0, no matter how small the neighborhoods, there are two non-specialized points P and Q which are joined by at least one extremal of type k on which $J<J_0$, then there will be at least one extremal joining P_0 to Q_0 on which there are at least k points and at most $k+m-1$ points conjugate to P_0.

With the aid of this principle we have been able, in the examples which are to follow, to give strong existence theorems which hold not only for non-specialized points, but also for specialized points as well.

7. *An Example.* Suppose we have an analytic surface S, with boundary b, and homeomorphic to a circular disc. On S consider the integral of arc length. Suppose S is extremal convex. Suppose S possesses a knob-shaped protuberance. More exactly suppose there is a portion of S homeomorphic to a disc, and bounded by a closed geodesic g that is shorter than nearby closed curves.

Any two points P and Q near g, or between g and b, will be connected by an infinite number of geodesics of type zero. By the aid of the preceding theorems one can then prove the following.

There will be an infinite number of geodesics joining P and Q upon each of which there is at least one point conjugate to P.

The later geodesics will include geodesics of arbitrarily great length which pass out of the region between g and b, and out of the neighborhood of g, thereby necessarily crossing the knob in some way.

This example suggests the more general question as to what effect the existence of a closed extremal in S will have upon the nature of extremals joining two points in S. If one admits closed extremals with conjugate points on them in varying numbers, and considers the problem in the different dimensions, one finds both curious and varied effects. The results are intimately connected with the so-called Poincaré rotation number.

8. *Extremals on General Regions.* The study of extremals in regions of general connectivities has been undertaken as yet only in a preliminary way. However, one case, namely the case of a region Σ homeomorphic with a portion of m-space between two concentric $(m-1)$-spheres has been studied, and has yielded much light. As in the case of the earlier region S so here we suppose Σ extremal-convex. A first result is the following.

If P and Q are any two non-specialized points on Σ, there exists *at least one extremal joining P to Q of every type k congruent to zero* mod $m-2$.

It is a curious fact that, while this theorem affirms the existence of an infinite number of extremals when $m>2$, it affirms the existence of but one extremal when $m=2$. However, the gap is filled by the affirmation that when $m=2$ there is always an infinite number of extremals joining P to Q of the one type zero.

Concerning any pair of points P and Q, whether specialized or not, we can prove the following.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-4$ *consecutive zeros* $(m>2)$.

In proving this theorem we have proved the a priori existence of an infinite set of extremals. This theorem shows that both the region and the dimension notably affect the results. It affords a contrast between the search for extremals that give minima and the search for extremals in general. Of the infinite set of extremals hereby affirmed to exist at most the first can give a minimum.

A simple example of an integral and extremal convex region such as Σ is the following. As the integral we take

$$\int \frac{1 + (r - 1)^2}{r} ds,$$

where ds is the differential of the arc length in m-space, and r the distance to the origin. The region between the two $(m-1)$-spheres, $r=a$ and $r=b$, is extremal-convex if we take $0<a<1<b$. It is easy to verify our theorems for any two points on the $(m-1)$-sphere, $r=1$.

9. *Extremals on Closed Manifolds.* Here the simplest and most important type of manifold has been investigated, namely an analytic manifold Σ homeomorphic to an m-sphere. Among others, the following two existence theorems have been proved.

If P and Q are any two non-specialized points on Σ*, there* *exists at least one extremal joining P to Q of every type k congruent to zero*, mod $m-1$.

In the conjugate number sequence for any two points P and Q whatsoever there can nowhere be more than $2m-3$ *consecutive zeros* $(m>1)$.

In proving the first of these theorems there is introduced a new method which might be called *the method of continuation of extremals by homeomorphisms*. First the existence of non-bounding r-families is gone into on the m-sphere. The greatest aids here are apparently the fundamental theorems on the characterization of conjugate points by deformations, and the knowledge of all the facts about geodesics on m-spheres. Once the existence of the non-bounding r-families is proved for the m-sphere it is of course proved for Σ. The existence of the required extremals then follows on Σ.

If Σ were any analytic closed manifold whatsoever, the method would require the selection of some manifold Σ_1 homeomorphic to Σ, upon which the study of extremals was as simple as possible. From facts about Σ_1 essentially belonging to the calculus of variations one would pass to facts about r-families, that is, facts essentially of analysis situs. From these last facts one would finally pass to the theorems about extremals on Σ.

This method may be contrasted with the much-used method of analytic continuation of extremals with respect to a parameter. Here one may be kept from the ultimate facts by the impossibility or difficulty of continuation beyond a certain point, or, in case one deals with an infinite set of extremals, with difficult questions of uniformity. All such difficulties as these are foreign to the above methods of continuation by homeomorphisms.

10. *The Case* $m=2$.* The case $m=2$ is essentially general in so far as deformations in the small are concerned so that the deformation theory given in the paper on the case $m=2$ needs practically no change to hold for the general case. A

* Morse II, loc. cit.

peculiarity of the case $m=2$ is that in an extremal convex region there can never be more than a finite number of extremals joining two points among the extremals with lengths less than a finite constant.

In the case $m=2$ it appeared desirable in a first paper, in the part that dealt with the theory in the large, to assume that the region was covered by a proper field of extremals, and to assume the extremals reversible. This had as a consequence that there were at most a finite set of extremals in S joining two points, but what was more important it reduced the essential passage from the small to the large to a matter of about three pages.* The problem of the extremals joining two fixed points when $m=2$ under this special field hypothesis has been taken up at the author's suggestion by D. E. Richmond† with a view to finding how complete were the relations already found. By methods peculiar to the special case treated he has established the completeness of the relations up to a certain point, and has found certain additional restrictive relations of a very elegant nature.

It is of interest to state in this connection that the author has proved his own relations between the numbers of critical points of a function and the connectivities of a general region are complete in general, and turning to the calculus of variations has proved his relations complete for the theory of extremals from a point to a manifold, and this under general hypotheses and for a general m. The method of proving a set of relations complete has been to take any set of integers M_k which satisfy the author's relations, and to set up an example to which these integers M_k belong.

11. *Closed Extremals.*‡ Closed extremals have been treated by the author up to the present time only for the case $m=2$.

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* Morse II, loc, cit., pp. 227–231.

† Richmond, *Number relations between types of extremals joining a pair of points*, American Journal of Mathematics, vol. 50 (1928), pp. 370–388; and *A new proof of certain relations of Morse in the calculus of variations in the large*, which will appear in this Bulletin, Mch.-Apr., 1929.

‡ Morse II, loc. cit. Also Hadamard, loc. cit.

Here we start by cutting across the given closed extremal g with n perpendiculars whose intersections with g follow each other cyclically, and are so numerous and so near together that the resulting n segments of g have no pairs of conjugate points on them. We let $(u_1, \cdots, u_n) = (u)$ be distances from g measured respectively along these n perpendiculars, and we set up the closed broken extremal E joining successively the points on these perpendiculars at the respective distances (u) from g. Let $J(u)$ be the value of J along E.

As previously $J(u)$ will have a critical point when $(u) = (0)$. But here the quadratic form H of the terms of second order of $J(u)$ in an expansion about $(u) = (0)$ has marked differences from the form associated with an extremal joining two fixed points. For the purposes of the following theorems suppose g mapped conformally on a segment g_1 of the x axis of length ω, and this by a transformation which has the period ω in x.

The rank of H is n, $n-1$, or $n-2$, according as the Jacobi differential equation corresponding to g_1 has respectively no solutions of period ω not identically zero, has a one-parameter family of solutions of period ω not identically zero, or has no solutions other than those of period ω.

When one comes to the type number of H one finds another difference from the earlier results in that this type number does not depend solely on the distribution of conjugate points. In case the rank of H is n the type number depends upon a further classification of closed extremals as follows.

If $x = x_0$ is not conjugate to $x_0+\omega$, then in the xy plane any point (x_0, a), $a > 0$, neighboring $(x_0, 0)$ can be joined to $(x_0+\omega, a)$ by an extremal segment g'. The segment of g from x_0 to $x_0+\omega$ will be said to be relatively *convex* or *concave* according as the slope of g' at $x_0+\omega$ is greater or less than the slope of g' at x_0. In case the rank of H is n these slopes will not be equal, and in this case the type number of H is now determined as follows.

Let m be the number of points conjugate to x_0 preceding

$x_0+\omega$. When x_0 is not conjugate to $x_0+\omega$, the type number of H is m if g is convex, and $m+1$ if g is concave. When x_0 is conjugate to $x_0+\omega$, the type number is m.

In case the rank of H is $n-1$ or $n-2$, and the given closed extremal is isolated, it can be shown that for all subsequent purposes g may be counted as equivalent to a finite set of closed extremals for which the rank of H is n.

Closed extremals were considered in the large in a ring-shaped region bounded by two simple closed curves. For such a region the theory in the large is not essentially different from that previously given and will not be elaborated here.

There remains the important study of closed extremals in the higher spaces,* and on closed, as well as open, manifolds. Here even the most casual inspection discloses a wide and varied store.

12. *Extremals through a Point O cut Transversally† by a Closed Manifold Σ.* We here make the same hypotheses with respect to the integral as in §1. In the domain of definition of F we suppose we have a closed, analytic, regular,‡ $(m-1)$-dimensional manifold Σ. We suppose also that Σ is so placed in the field of extremals through O that there are no conjugate points to O on the extremals from O to Σ until after their intersections with Σ.

Let g be an extremal through O cut transversally by Σ at P. Let (v) be a set of $m-1$ parameters in an admissible‡

* For a first remarkable difference see Carathéodory, *Über geschlossene Extremalen und periodische Variationsprobleme in der Ebene und im Raume*, Annali di Matematica, (4), vol. 2 (1925), pp. 297–320. Here among other things it is shown that the Poincaré Law does not hold in the higher spaces.

† An interesting paper in this connection is that by White, M. B., *The dependence of focal points upon curvature for problems of the calculus of variations in space*, Transactions of this Society, vol. 13 (1912), pp. 175–198. This paper is primarily concerned with the differential and geometric nature of focal points in the small, in three-space, in particular in connection with the search for minima. No use is made of these results by the author.

‡ An analytic manifold will be called regular if on it the coordinates (x) of points neighboring any given point admit a representation in terms of $m-1$ parameters in which not all of the jacobians of $m-1$ of the co-

representation of Σ near P. On the extremals near g cut transversally by Σ at points (v), the coordinates (x) will be analytic functions of the parameters (v), and of the arc length s measured on these extremals from Σ. Let $\Delta(s)$ be the jacobian of these m coordinates (x) with respect to (v) and s, evaluated for the set $(v)=(v)_0$ that corresponds to P.

The points on g different from P at which $\Delta(s)=0$ *are called the focal points of P on g, and the order of vanishing of* $\Delta(s)$ *at these points the order of the focal points.*

The value of J taken from O to the points (v) will be indicated by $J(v)$. It appears first that $J(v)$ has a critical point when $(v)=(v)_0$. Let H be the quadratic form of the terms of second order in the expansion of $J(v)$ about $(v)=(v)_0$. Set $m-1=n$. We have the following theorem.

If O is a focal point of P of the rth order, the rank of the matrix of the form H is $n-r$. *If O is not a focal point of P, the type number of H equals the number of focal points of P between P and O, always counting focal points according to their orders.*

From the author's relations* between the critical points of a function we then obtain the following theorem in the large.

Suppose O is not a focal point of Σ, *nor on* Σ. *Of the extremals from O to* Σ *which are cut transversally by* Σ *at their ends P on* Σ, *let* M_i *be the number upon which there are i focal points of P between O and P. Then between the numbers* M_i *and the connectivities* R_i *of* Σ *the following relations hold:*

$$
\begin{aligned}
M_0 &\geqq 1+(R_0-1),\\
M_0-M_1 &\leqq 1+(R_0-1)-(R_1-1),\\
M_0-M_1+M_2 &\geqq 1+(R_0-1)-(R_1-1)+(R_2-1),\\
&\cdots\cdots\cdots\cdots\\
M_0-M_1+\cdots+(-1)^nM_n &\\
&= 1+(R_0-1)-+\cdots+(-1)^n(R_n-1).
\end{aligned}
\tag{5}
$$

ordinates with respect to the $m-1$ parameters are zero at once. Such a representation is called admissible.

* See Morse I, and Morse, *The analysis and analysis situs of regular n-spreads in* $(n+r)$*-space*, Proceedings of the National Academy, vol. 13 (1927), pp. 813–817.

If O is on Σ the above relations still hold if we count O as a nul-extremal of type zero.

The following corollary furnishes a strong existence theorem.

If O is not a focal point of Σ, the number of extremals from O cut transversally by Σ on which there are k focal points of Σ, is at least $R_k - 1$. *The total number of extremals is at least*

$$(6) \qquad 1 + (R_0 - 1) + \cdots + (R_n - 1).$$

The theorem and corollary apply at once to the problem of the number of normals from O to Σ, as follows.

If O is not a center of principal normal curvature of Σ, and if M_i *is the number of normals from O to Σ on which there are i centers of principal normal curvature of Σ between O and the feet of the normals on Σ, then the relations* (5) *hold.*

For example, if Σ is an admissible surface homeomorphic with a torus it follows from (6) there will be at least four normals to Σ from any point not a center of principal normal curvature. If Σ is a surface in 4-space homeomorphic with the complex obtained by identifying the opposite faces of an ordinary cube, one sees from (6) that there will be at least eight normals from a point not a center of principal normal curvature. It is possible to multiply these examples in any space. Moreover, even when O is a center of principal normal curvature, it is still possible to get strong existence theorems by a limiting process.

The question of the number of $(m-1)$-planes tangent to Σ through a given $(m-2)$-plane is closely related to the preceding theory on normals, and it is easy to proceed to existence theorems on the tangency of this $(m-1)$-plane. In fact there is a whole theory of the relations of straight lines, hyperplanes and hyperspheres to closed manifolds that can be treated after the spirit of this section.

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