ON UNIFORM SUMMABILITY OF SEQUENCES OF CONTINUOUS FUNCTIONS*

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In a paper by D. C. Gillespie and W. A. Hurwitz,† it was shown that any bounded real sequence $\{s_n(x)\}$ of continuous functions, defined over a closed compact set A in a metric space, which converges over A to a continuous function $s(x)$, is uniformly summable to $s(x)$ by a regular transformation. The transformations used were transformations with square matrices which were constructed in terms of the elements of the $\{s_n(x)\}$ sequences considered; hence different transformations were used to effect the uniform summability of different sequences. In §8 the question was raised as to whether some restricted class of regular transformations of a familiar type might serve for the uniform summability of all sequences of the class considered; the question was partially answered by exhibiting a single sequence for which the Cesàro, Euler-Abel, and Borel transformations will not suffice. It is the object of this note to show that also no one regular transformation whatever of the general form considered below will suffice.

Nearly every transformation that has been considered in the theory of summability‡ is a special case of the following one. Let T and A be sets in metric spaces, let T have a limit point t_0 not belonging to T, and let the functions $a_k(t)$, $k=1, 2, 3, \cdots$ be defined over T. Then if a sequence $\{s_n(x)\}$, defined over A, is such that

$$(G) \qquad \sigma(t, x) = \sum_{k=1}^{\infty} a_k(t) s_k(x)$$

converges for all t in T and x in A and if

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‡ For a bibliography of the subject, see L. L. Smail, *History and Synopsis of the Theory of Summable Infinite Processes*, University of Oregon Press, 1925.

$$\lim_{t\to t_0(T)} \sigma(t, x) = \sigma(x), \tag{1}$$

the sequence $\{s_n(x)\}$ is said to be summable (G) to $\sigma(x)$; if the limit is approached uniformly with respect to x in (1), then $\{s_n(x)\}$ is said to be uniformly summable (G) to $\sigma(x)$.

The answer to the question we are considering is contained in the following theorem.

THEOREM 1. *In order that (T) may be such that*

$$\lim_{t\to t_0(T)} \sigma(t, x) = 0, \qquad \textit{uniformly with respect to } x \tag{2}$$

*for every sequence $\{s_n(x)\}$ of continuous functions, defined over an infinite set A, such that $s_n(x)$ is bounded over A for all n and $\lim_{n\to\infty} s_n(x) = 0$ over A, it is necessary and sufficient that**

$$\lim_{t\to t_0(T)} \sum_{k=1}^{\infty} | a_k(t) | = 0. \tag{3}$$

To establish necessity, let (G) be a given transformation which implies (2) for every admissible sequence $\{s_n(x)\}$. Considering, for each positive integer k, the sequence $s_n(x) = 0$, $n \neq k$, and $s_k(x) = 1$, we see that (G) must satisfy the condition

$$\text{for each } k, \quad \lim_{t\to t_0(T)} a_k(t) = 0. \tag{4}$$

We shall show that (3) is a necessary condition by supposing that (G) satisfies (4) but not (3) and defining an admissible sequence $\{s_n(x)\}$ for which (2) fails. Since (3) is denied there is a number $\theta > 0$ such that

$$\limsup_{t\to t_0(T)} \sum_{k=1}^{\infty} | a_k(t) | > \theta;$$

hence there is a sequence $\{t_n\}$ of points of T with the limit t_0 such that

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} | a_k(t_n) | > \theta. \tag{5}$$

Using (5), choose an index n_1 such that

* A transformation (G) which satisfies (3) is not regular; it is a null transformation.

$$\sum_{k=1}^{\infty} | a_k(t_{n_1}) | > \theta$$

and choose $N_1 > n_1$ such that

$$\sum_{k=1}^{N_1} | a_k(t_{n_1}) | > \theta.$$

Using (4) and (5), choose $n_2 > N_1$ such that

$$\sum_{k=1}^{N_1} | a_k(t_\alpha) | < \theta/2 \text{ for } \alpha \geqq n_2 \text{ and } \sum_{k=1}^{\infty} | a_k(t_{n_2}) | > \theta,$$

and choose $N_2 > n_2$ such that

$$\sum_{k=1}^{N_2} | a_k(t_{n_2}) | > \theta.$$

Proceeding in this manner, choose $n_1 < N_1 < n_2 < N_2 < n_3 < \cdots$ such that for $p = 2, 3, 4, \cdots$

$$\sum_{k=1}^{N_{p-1}} | a_k(t_\alpha) | < \theta/2 \text{ for } \alpha \geqq n_p \text{ and } \sum_{k=1}^{N_p} | a_k(t_{n_p}) | > \theta.$$

Then, defining $N_0 = 0$, we have

$$(6) \qquad \sum_{k=N_{p-1}+1}^{N_p} | a_k(t_{n_p}) | > \frac{\theta}{2}, \quad p = 1, 2, 3, \cdots.$$

Let $\{x_p\}$ be a sequence of distinct points of A such that no point of $\{x_p\}$ is a limit point of $\{x_p\}$. Then corresponding to each point x_p of $\{x_p\}$ there is a positive number r_p such that $r(x_p, x_q) > 2r_p$* for $q \neq p$; let the set of points x of A for which $r(x_p, x) < r_p$ be denoted by A_p. Then $A_1, A_2, A_3, \cdots$ are mutually exclusive subsets of A. Define the sequence $\{s_p(x)\}$ over A as follows: For k such that $N_{p-1} < k \leqq N_p$ and for $p = 1, 2, 3, \cdots$

$$s_k(x) = \begin{cases} [1 - r(x_p, x)/r_p] \operatorname{sgn} a_k(t_{n_p}) & \text{over } A_p, \\ 0 & \text{over } A - A_p. \end{cases}$$

* $r(a_1, a_2)$ is used to denote the distance between two points a_1 and a_2 of A.

It is readily seen that $s_n(x)$ is continuous over A for each n, $s_n(x)$ is bounded over A for all n and $\lim s_n(x)=0$ over A; hence $\{s_n(x)\}$ is an admissible sequence. But for each index p

$$s_k(x_p) = \begin{cases} \operatorname{sgn} a_k(t_{n_p}) & \text{for } N_{p-1} < k \leqq N_p, \\ 0 & \text{for } k \leqq N_{p-1} \text{ and } k > N_p, \end{cases}$$

and hence for each index p

$$\sigma(t_{n_p}, x_p) = \sum_{k=N_{p-1}+1}^{N_p} a_k(t_{n_p}) \operatorname{sgn} a_k(t_{n_p}) = \sum_{k=N_{p-1}+1}^{N_p} | a_k(t_{n_p}) | .$$

Using (6) and the preceding relation we obtain

$$\limsup_{p\to\infty} \sigma(t_{n_p}, x_p) \geqq \theta/2;$$

thus (2) is denied and the necessity of (3) is established.

The sufficiency of (3) is readily established by proving the following theorem.

THEOREM 2. *In order that* (G) *may be such that* (2) *is implied for every bounded sequence of functions*, (3) *is sufficient.*

Let (G) satisfy (3) and let M be a constant such that $|s_n(x)| < M$ over A for all n. Then

$$| \sigma(t, x) | \leqq \sum_{k=1}^{\infty} | a_k(t) | | s_k(x) | \leqq M \sum_{k=1}^{\infty} | a_k(t) |$$

and (2) follows.

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