From the above and from (8), we obtain

$$\left\{\frac{1-\alpha}{r}\right\} = \left\{\frac{\alpha}{r}\right\},$$

and from (7) it follows that

$$r^{p-1} \equiv 1 \pmod{p^2}.$$

As before, a similar proof obtains when y is divisible by p.

OTTAWA, CANADA

ON THE SOLUTION OF THE EULER EQUATIONS FOR THEIR HIGHEST DERIVATIVES*

BY H. V. CRAIG

1. *Introduction.* J. H. Taylor† has given two elegant methods of solving for their highest derivatives the Euler equations associated with the integral $\int F(x, \dot{x})dt$. In this paper these two methods are modified so as to apply to the more general case in which the Euler equations contain derivatives of order higher than the second.

2. *Notation.* Throughout this paper we shall employ vector notation and shall use dots and enclosed superscripts to indicate differentiation with respect to the parameter. Thus x, $\dot{x}$, $x^{(m)}$ will stand for the sets

$$x^1, x^2, \cdots, x^n; \frac{dx'}{dt}, \frac{dx^2}{dt}, \cdots, \frac{dx^n}{dt}; \frac{d^m x^1}{dt^m}, \frac{d^m x^2}{dt^m}, \cdots, \frac{d^m x^n}{dt^m},$$

respectively. Partial derivatives will be denoted by means of subscripts, thus

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† J. H. Taylor, *The reduction of Euler's equations to a canonical form*, this Bulletin, vol. 31 (1925) p. 257.

$$\frac{\partial F(x,\dot{x},\cdots,x^{(m)})}{\partial x^i} = F_{i^0};\qquad \frac{\partial F(x,\dot{x},\cdots,x^{(m)})}{\partial \dot{x}^i} = F_{i^1};$$

$$\frac{\partial F(x,\dot{x},\cdots,x^{(m)})}{\partial \ddot{x}^i} = F_{i^2};\ \cdots .$$

However, if the differentiation is with respect to the highest derivatives present we shall further abbreviate by omitting the m, that is,

$$\frac{\partial F(x,\dot{x},\cdots,x^{(m)})}{\partial x^{i(m)}} = F_i .$$

Summations are to be understood when repeated indices occur.

3. *The Calculus of Variations Problem and the Euler Equations.* We consider a function $F(x, \dot{x}, \cdots, x^{(m)})$, $m > 1$, with properties to be specified and seek among all curves of class $2m$ lying in a certain region of an n-space and satisfying certain boundary conditions, the one which gives the integral

$$I = \int_{t_1}^{t_2} F(x,\dot{x},\cdots,x^{(m)})dt$$

its minimum value.

As a first hypothesis on F we suppose, as just implied, that the solution of this problem exists uniquely. The additional hypotheses are: (a), that F is of class $m+1$; (b), that the classical $\overline{F}_1$ function* associated with F does not vanish along the solution; and (c), that I is independent of the choice of the parameter.

Zermelo has shown† that this invariance of I implies the following identities in $x, \dot{x}, \cdots, x^{(m)}$:

(1) $\qquad \dot{x}^i F_i \equiv 0,\ \ i$ ranges from 1 to n,

(2) $\qquad x^{i(\alpha)} E_{\alpha i} \equiv F,\ \ \alpha$ ranges from 1 to m,

* We have used $\overline{F}_1$ to avoid confusion with $F_1 = \partial F/\partial x^{1(m)}$. For a discussion of the $\overline{F}_1$ function, see Oscar Bolza, *Vorlesungen über Variationsrechnung*, Leipzig, Teubner, 1909, p. 13. Oscar Bolza, *Lehrbuch der Variationsrechnung*, p. 196.

† See Adolph Kneser, *Lehrbuch der Variationsrechnung*, Leipzig, Teubner, 1925, p. 217.

where E_{qi} is defined by the equation

$$E_{qi} \equiv (-1)^{\beta} F^{\beta}{}_{i}{}^{q+\beta}, \quad (\beta \text{ ranges from } 0 \text{ to } m-q).$$

We observe from this definition that $E_{0i}=0$, $i=1, 2, \cdots, n$, are the Euler equations associated with F. These equations are not independent, but satisfy the relation

$$\dot{x}^i E_{0i} = 0. \tag{3}$$

This may be established as follows. Differentiating (2) we obtain

$$\dot{F} = x^{i(\alpha)} \dot{E}_{\alpha i} + x^{i(\alpha+1)} E_{\alpha i}.$$

If in this we replace the second term by its value as given by the formula

$$E_{qi} = F_{iq} - E_{q+1\, i}$$

the relation becomes

$$\dot{F} = x^{i(\alpha)} [F_i^{\alpha-1} - E_{\alpha-1\, i}] + x^{i(\alpha+1)} E_{\alpha i}.$$

Since $E_{mi} = F_i$, this may finally be written

$$\dot{F} = x^{i(\alpha)} F_i^{\alpha-1} + x^{i(m+1)} F_i - \dot{x}^i E_{0i} + x^{i(\alpha)} E_{\alpha-1i} - x^{i(\alpha)} E_{\alpha-1i};$$

hence $\dot{x}^i E_{0i}=0$.

Since (1) evidently leads to

$$\dot{x}^i F_{ij} = 0, \qquad (j = 1, 2, \cdots, n), \tag{4}$$

it follows that the determinant $|F_{ij}|$ (and this is the determinant of the coefficients of $x^{(2m)}$ in the Euler equations) vanishes. Accordingly, the problem of solving the Euler equations for their highest derivatives requires special consideration. We shall make it a part of our hypotheses on F that the rank of this determinant be $n-1$.

For use in determining the rank of certain determinants which will appear presently, we insert here a few miscellaneous observations. As a consequence of equation (4) and the rank of the determinant $|F_{ij}|$, the cofactors F^{ij} of the latter satisfy the following relations:

$$\frac{\dot{x}^1}{F^{i1}} = \frac{\dot{x}^2}{F^{i2}} = \cdots = \frac{\dot{x}^n}{F^{in}}.$$

If we note that the quantities F^{ii} are symmetric in their indices, these equalities are seen to be expressible in the form

$$\frac{\dot{x}^i\dot{x}^j}{F^{ij}} = \frac{\dot{x}^k\dot{x}^l}{F^{kl}}, \tag{5}$$

where i, j, k, l may each be any number of the set $1, 2, \cdots, n$ and no summation is to be understood. The reciprocal of the common value of the members of (5) is the $\overline{F}_1$ function of our problem.

4. *The First Method of Solution.** Let $H(x, \dot{x})$ be any function of class $2m-1$, homogeneous of degree plus one in $\dot{x}$, and non-vanishing along the solution of our problem. With these restrictions we may so select the parameter that H will maintain the value unity along the solution. Differentiating the equation $H=1$, $2m-1$ times with respect to t, yields the relation

$$x^{i(2m)}H_i + r = 0. \tag{6}$$

Here we have written explicitly only the terms in $x^{(2m)}$ and have represented by r the remaining terms. This equation we adjoin to the system

$$x^{i(2m)}F_{ij} + wH_j + R_j = 0, \tag{7}$$

R_j being so chosen that these relations reduce to the Euler equations for $w=0$. The system (6, 7) is linear in the variables $x^{(2m)}$ and w, has a non-vanishing determinant, and determines the same set of values for $x^{(2m)}$ as the Euler equations. To prove this last statement we multiply the equations of the set (7) by $\dot{x}^j$ and sum. Because of (3) and the conditions imposed on H (the homogeneity of H implies $\dot{x}^iH_i=H$) the result is $w=0$. The determinant of the system (6, 7) is equal to

$$\pm H_iH_jF^{ij} = \pm \overline{F}_1\dot{x}^i\dot{x}^jH_iH_j = \pm \overline{F}_1 \neq 0.$$

5. *The Second Method of Solution.* Let us replace the function $F(x, \dot{x}, \cdots, x^{(m)})$ of our calculus of variations problem with a new function $f(x, \dot{x}, \cdots, x^{(m)})$, which we define as follows:

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* Taylor points out the incidence of this method in an article entitled *The properties of curves in space which minimize a definite integral*, by Mason and Bliss, Transactions of this Society, vol. 9 (1908), p. 443.

$$f(x,\dot{x},\cdots,x^{(m)}) \equiv F(x,\dot{x},\cdots,x^{(m)}) + \frac{1}{2}\left\{\frac{d^{m-1}H(x,\dot{x})}{dt^{m-1}}\right\}^2.$$

We restrict the function H as in the preceding section and select the parameter as before so that H maintains the value unity along the unique solution C.

For such a parameter it is evident that

$$\int_C f(x,\dot{x},\cdots,x^{(m)})dt = \int_C F(x,\dot{x},\cdots,x^{(m)})dt,$$

and

$$\int_{\bar{C}} fdt > \int_C Fdt,$$

if $\bar{C}$ is any other admissible curve. Therefore the curve C is also an extremal of the integral $\int fdt$. (The problem associated with f is not a Weierstrass problem since t has a special meaning.) Furthermore the determinant $|f_{ij}|$ $(f_{ij}=F_{ij}+H_iH_j)$ is different from zero since

$$|f_{ij}| = H_iH_jF^{ij} = \bar{F}_1\dot{x}^i\dot{x}^jH_iH_j \neq 0.*$$

Hence the values of the quantities $x^{(2m)}$ along any extremal may be obtained by solving the Euler equations associated with f by Cramer's rule.

It can be shown that the left members of the Euler equations in the unsolved form are the components of a covariant vector. The method of solution outlined above gives us a simple contravariant description of this vector.

THE UNIVERSITY OF TEXAS

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* See J. H. Taylor, loc. cit. p. 261, for the development of a similar determinant.