A SET OF CYCLICLY RELATED FUNCTIONAL EQUATIONS*

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An examination of the first-order differential equation $y' = p(x)y$ and the second-order system $y' = p(x)y + q(x)z$, $z' = p(x)z + q(x)y$, where $p(x)$ and $q(x)$ are integrable functions of the real variable x, shows that their general solutions may be expressed in exponential form. Conversely, the solutions of the first-order equation may be used to define the exponential function and the solutions of the second-order system may be used to define the circular and hyperbolic functions as well as to give the relations that exist between these two sets of functions. These facts lead one to consider the general system

$$(1) \qquad y_i' = \sum_{k=1}^{n} A_k(x) y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where n is a positive integer, m and h are integers or zero, and where the coefficients $A_k(x)$ are L-integrable functions of x on an interval of definition X. More generally, one is led to consider the functional system

$$(2) \qquad L(y_i) = \sum_{k=1}^{n} A_k y_{i+m+hk}, \quad (i = 1, \cdots, n;\ y_{j+n} \equiv y_j),$$

where the functional operator L has the property $L(ay+bz) \equiv aL(y) + bL(z)$ for any constants a and b, and where the coefficients A_k are functions of a finite or countably infinite set of variables $(x_1, x_2, \cdots)$ in a domain D of these variables. It is to be noted that the operator L may combine partial differential operators of various orders, simple and multiple integrals, and many other operators, so that system (1) occurs as a very special case of system (2). The present paper considers system (2). As a special case of the results of the paper, the general solution of system (1) is obtained and its exponential character exhibited.

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THEOREM 1. *Let p and q be any positive integers* such that $pq=n$; then there exists a non-singular linear transformation carrying $y_1, \cdots, y_n$ into u_{it}, $i=1, \cdots, q$, $t=1, \cdots, p$, in such a way that system (2) goes into p independent systems each one of which consists of q equations involving q of the quantities u_{it} and no two of the systems have a u_{it} in common.*†

PROOF. Let $r_1, \cdots, r_p$ be the distinct pth roots of unity and let

$$(3)\quad u_{it} = \sum_{j=1}^{p}(r_t)^j y_{i+(j-1)q}, \quad (i = 1, \cdots, n, \cdots; t = 1, \cdots, p).$$

LEMMA. *$u_{it}=r_t u_{i+q,t}$, hence $u_{i+n,t}=u_{i+pq,t}=(r_t)^p$, $u_{it}=u_{it}$.*

Proof of Lemma. From (3), we have

$$r_t u_{i+q,t} = \sum_{j=1}^{p}(r_t)^{j+1}y_{i+jq} = \sum_{s=2}^{p}(r_t)^s y_{i+(s-1)q} + (r_t)y_i$$

$$= \sum_{s=1}^{p}(r_t)^s y_{i+(s-1)q} = u_{it}.$$

Proof of Theorem 1. Apply L to u_{it} as given by (3) and make use of (2):

$$L(u_{it}) = L\left(\sum_{j=1}^{p}[r_t]^j y_{i+(j-1)q}\right) = \sum_{j=1}^{p}[r_t]^j L(y_{i+(j-1)q})$$

$$= \sum_{j=1}^{p}[r_t]^j \sum_{k=1}^{n} A_k y_{i+m+hk+(j-1)q} = \sum_{k=1}^{n} A_k u_{i+m+hk,t}$$

$$= \sum_{k=1}^{q}\sum_{j=1}^{p} A_{k+(j-1)q} u_{i+m+hk+h(j-1)q,t}.$$

If we now apply the lemma, we obtain

$$(S_t)\quad L(u_{it}) = \sum_{k=1}^{q} B_{kt} u_{i+m+hk,t}, \quad (i = 1, 2, \cdots, q;\ r_t u_{i+q,t} = u_{it}),$$

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* One such pair being n and 1.

† We assume the existence of a set of functions $y_1, \cdots, y_n$ such that (2) holds almost everywhere in D.

where

$$B_{kt} = \sum_{j=1}^{p} [r_t]^{h(1-j)} A_{k+(j-1)q}.$$

System (S_t) consists of q equations involving $u_{1t}, \cdots, u_{qt}$. The systems $(S_1), (S_2), \cdots, (S_p)$ are the ones required for Theorem 1. It remains to prove that the transformation (3) is non-singular. When i is fixed, a determinant of coefficients of $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$ in (3) is

$$D_i = \begin{vmatrix} r_1 & r_1^2 & \cdots & r_1^p \\ r_2 & r_2^2 & \cdots & r_2^p \\ \cdot & \cdot & \cdot & \cdot \\ r_p & r_p^2 & \cdots & r_p^p \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots 1 & 1 \\ r_2 & r_2^2 & \cdots (r_2)^{p-1} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ r_p & r_p^2 & \cdots (r_p)^{p-1} & 1 \end{vmatrix}.$$

D_i is recognized as a Vandermonde determinant and has the value

$$\prod_{\substack{k,j=1 \\ j>k}}^{p} (r_j - r_k).$$

A determinant of the transformation (3) is therefore

$$D = \prod_{1}^{q} D_i = (D_i)^q \neq 0.$$

The quantities $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$ may be thought of as the roots of an equation and, with this interpretation, the quantity u_{it} becomes a Lagrange resolvent.* Thus $u_{it} = [r_t, Y_i]$, where r_t is any pth root of unity and where the components of Y_i are $y_i, y_{i+q}, \cdots, y_{i+(p-1)q}$. This interpretation is quite helpful and enables one to solve equations (3) for $y_1, \cdots, y_n$ with ease. By a well known property of the Lagrange resolvent,† we have

$$\sum_{t=1}^{p} (r_t)^{-k} u_{it} = p y_{i+(k-1)q}, \quad (i = 1, \cdots, q; \ k = 1, \cdots, p).$$

* See Cajori, *Theory of Equations*, New York, Macmillan, 1904, pp. 129–133.

† See Cajori, loc. cit., p. 130.

Hence

$$y_{i+(k-1)q} = \left[\sum_{t=1}^{p} (r_t)^{-k} u_{it} \right] \Big/ p, \tag{4}$$

$$(i = 1, \cdots, q;\ k = 1, \cdots, p).$$

The special case of transformation (3) where $p=n$ and $q=1$ is important since it resolves system (2) into n equations, each of which contains a single function u_{it}. For a wide class of operators L, it will then be possible to solve these n equations for the u's and thus obtain $y_1, \cdots, y_n$ by means of equations (4). Thus system (1) yields $u_t' = B_t r_t^{-(m+h)} u_t$ (dropping the subscript 1), $(t=1, \cdots, n)$. Hence we obtain $u_t(x) = c_t e^{f(t,x)}$, where $c_1, \cdots, c_n$ are arbitrary constants and

$$f(t, x) = \sum_{j=1}^{n} (r_t)^{-m-jh} \int_a^x A_j(s) ds. \tag{5}$$

The general solution of system (1) is

$$y_k(x) = \sum_{t=1}^{n} (r_t)^{-k} C_t e^{f(t,x)}, \quad (k = 1, \cdots, n), \tag{6}$$

where $f(t,x)$ is given by (5) and $C_1, \cdots, C_n$ are arbitrary constants. Formulas (6) show the exponential character of the solutions of all systems of type (1) and these differential systems are seen to define a class of functions which have many properties in common with the exponential, sine, and cosine functions. These properties are more evident when the general solution of (5) is put in trigonometric form. We now obtain this form.

Let the notation be chosen so that

$$r_t = \cos(2t\pi/n) + i \sin(2t\pi/n).$$

We have $r_t = (r_1)^t$ and $f(t,x)$ has the form

$$f(t, x) = \sum_{j=1}^{n} r_1^{-t(m+jh)} \int_a^x A_j(s) ds,$$

while

$$f(n - t, x) = \sum_{j=1}^{n} r_1^{t(m+jh)} \int_a^x A_j(s) ds.$$

Let

$$(7)\qquad P(t,\, x) = \sum_{j=1}^{n} \int_a^x A_j(s)ds \cos\,(2t\pi(m + jh)/n),$$

$$(8)\qquad Q(t,\, x) = -\sum_{j=1}^{n} \int_a^x A_j(s)ds \sin\,(2t\pi(m + jh)/n),$$

then $f(t,x) = P(t,x) + iQ(t,x)$ and $f(n-t,x) = P(t,x) - iQ(t,x)$.

Odd-Order Case. Let $n = 2d+1$, where d is an integer. We have

$$y_k(x) = \sum_{p=1}^{d} \left[C_p r_1^{-pk} e^{f(p,x)} + C_{n-p} r_1^{-(n-p)k} e^{f(n-p,x)}\right] + C_n e^{f(n,x)}.$$

Upon expressing $f(p,x)$ and $f(n-p,\, x)$ in terms of P and Q and grouping the terms of $y_k(x)$, we obtain

$$(9)\quad y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} e^{P(p,x)} \left[C_p e^{Qi} + C_{n-p} e^{-Qi}\right] \cos\,(2pk\pi/n)$$
$$+\, i\left[-\,C_p e^{Qi} + C_{n-p} e^{-Qi}\right] \sin\,(2pk\pi/n)\,.$$

Let $y_{1k}(x)$ denote the $y_k(x)$ obtained from (9) when $C_p = C_{n-p} = H_p/2$ and let y_{2k} be the y_k obtained when $C_p = -C_{n-p} = K_p/(2i)$. We note that $y_k = y_{1k}(x) + y_{2k}(x)$, $(k = 1, \cdots, n)$, is a solution of system (1). Furthermore, this is the general solution of that system. We have

$$(10)\; y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} e^{P(p,x)} \left\{H_p \cos\left[Q(p,\, x) - 2pk\pi/n\right]\right.$$
$$\left. +\, K_p \sin\left[Q(p,\, x) - 2pk\pi/n\right]\right\},$$

where C_n, H_p, K_p, $p = 1, \cdots, d$, are arbitrary constants and $f(p,x)$, $P(p,x)$, $Q(p,x)$ are given by formulas (5), (7), and (8). If we let

$$N_p = [H_p^2 + K_p^2]^{1/2}, \quad H_p/N_p = \cos M_p, \quad -K_p/N_p = \sin M_p,$$

formulas (10) become

$$(11)\; y_k(x) = C_n e^{f(n,x)} + \sum_{p=1}^{d} N_p e^{P(p,x)} \cos\left[Q(p,\, x) + M_p - 2pk\pi/n\right],$$

where C_n, N_p, M_p, $p = 1, \cdots, d$, are arbitrary constants.

Even-Order Case. In the case $n=2d$, d an integer, formula (6) can be put in the form

$$(12)\qquad y_k(x) = \sum_{p=1}^{d-1}[C_p r_1^{-pk} e^{f(p,x)} + C_{n-p} r_1^{pk} e^{f(n-p,x)}] + C_n e^{f(n,x)} + C_d e^{f(d,x)} r_1^{-kd}.$$

Procedure entirely analogous to that used in treating the odd-order case leads to the following two forms of the general solution of system (1) in the case where n is an even integer:

$$(13)\quad y_k(x) = Ke^{f(n,x)} + He^{f(d,x)}[-1]^k + \sum_{p=1}^{d-1}[H_p \cos\{Q(p, x) - 2pk\pi/n\} + K_p \sin\{Q(p, x) - 2pk\pi/n\}]e^{P(p,x)},$$

$$(14)\quad y_k(x) = Ke^{f(n,x)} + He^{f(d,x)}[-1]^k + \sum_{p=1}^{d-1} N_p e^{P(p,x)} \cos[Q(p, x) + M_p - 2pk\pi/n],$$

where H, K, H_p, K_p, N_p, M_p, $p=1, \cdots, d-1$, are arbitrary constants and $f(p,x)$, $P(p,x)$, $Q(p,x)$ are given by formulas (5), (7), and (8).

It would be of interest to investigate specific properties of the solutions given by (10), (11), (13), and (14), for example the distribution of the roots of the solution functions.*

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* See L. E. Ward, American Mathematical Monthly, vol. 34 (1927), pp. 301–303, for a special third-order case.