As *dual* partial correlation coefficient we might take the invariant numbers

$$\rho_{jk} = \frac{\sigma_{jk}}{(\sigma_{jj})^{1/2}(\sigma_{kk})^{1/2}}.$$

All invariant functions of the quantities σ_{jk} are also functions of the quantities ρ_{jk} only. But the quantities ρ_{jk} have not the simple relation to the regression hyperplanes.

In problems of type B in more variables the symmetry axes of the correlation quadric come into consideration.

In problems of type C the regression planes and the correlation coefficients lose their sense, but not the symmetry axes. Here the theory becomes the well known theory of quadratic matrices under orthogonal substitutions with the unessential modification that similarity transformations are also permitted.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

NOTE ON THE EXISTENCE OF A POSITIVE FUNCTION ORTHOGONAL TO A GIVEN SET OF FUNCTIONS*

BY N. H. McCOY†

Let the finite set of functions:

$$\{f_j(x)\}: \qquad f_1(x), f_2(x), \cdots, f_m(x)$$

be continuous and linearly independent on the closed interval X, $(a \leqq x \leqq b)$. With reference to this set of functions, L. L. Dines‡ has shown the equivalence of the following properties:

(A) *Every linear combination of the functions changes sign on* X.

(B) *There exists a positive continuous function orthogonal to each function of the set on* X.

A sufficient condition for the set $\{f_j(x)\}$ to have properties (A) and (B) has also been given by Dines.§ It is in a form

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† National Research Fellow.

‡ *A theorem on orthogonal functions with an application to integral inequalities*, Transactions of this Society, vol. 30 (1928), pp. 425–438.

§ *On completely signed sets of functions*, Annals of Mathematics, vol. 28 (1926), pp. 393–395.

easily recognized when it is satisfied and in this case the positive function whose existence is asserted by (B) may be determined without difficulty. The primary purpose of this note is to give two other conditions of a different type either of which is sufficient to assure the presence of properties (A) and (B).

Let

$$F_i^{(1)}(x)=\int_a^x f_i(x)dx,\ F_i^{(k)}(x)=\int_a^x F_i^{(k-1)}(x)dx,\quad \begin{matrix}(i=1,2,\cdots,m),\\(k=2,3,\cdots,m),\end{matrix}$$

and consider the matrix,

$$\left\|\begin{matrix} f_1(x) & F_1^{(1)}(x)\cdots F_1^{(m)}(x)\\ f_2(x) & F_2^{(1)}(x)\cdots F_2^{(m)}(x)\\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ f_m(x) & F_m^{(1)}(x)\cdots F_m^{(m)}(x)\end{matrix}\right\|.$$

Denote by $(-1)^{k+1}D_k(x)$ the determinant of the matrix obtained from this matrix by striking out the kth column. We shall prove the following theorem.

THEOREM 1. *If there exists a value of x on X for which $D_1(x)$, $D_2(x),\cdots,D_{m+1}(x)$ are all different from zero and of the same sign, then the set $\{f_i(x)\}$ has properties* (A) *and* (B).

Let x_1 be a value of x satisfying the condition of the theorem. Suppose property (A) is not present. There then exists a set of real constants $c_i(i=1,2,\cdots,m)$ not all zero such that

$$c_1f_1(x)+c_2f_2(x)+\cdots+c_mf_m(x)>'0,\qquad(x\text{ on }X),\tag{1}$$

where the symbol $>'$ is to be read "is somewhere greater than and nowhere less than." The function on the left of (1) is a non-negative function of x and if we integrate it from a to x we get a function with the same property. That is,

$$c_1F_1^{(1)}(x)+c_2F_2^{(1)}(x)+\cdots+c_mF_m^{(1)}(x)>'0,\ (x\text{ on }X).\tag{2}$$

By repeating this process of integration we find,

$$\begin{matrix}c_1F_1^{(2)}(x)+c_2F_2^{(2)}(x)+\cdots+c_mF_m^{(2)}(x)>'0,\\ \cdot\\ c_1F_1^{(m)}(x)+c_2F_2^{(m)}(x)+\cdots+c_mF_m^{(m)}(x)>'0,\end{matrix}\qquad(x\text{ on }X).\tag{3}$$

If we substitute the particular value $x=x_1$ in (1), (2) and (3), we have

$$\sum_{j=1}^{m} \mathbf{c}_j F_j^{(k)}(x_1) \geqq 0, \qquad (k = 0, 1, 2, \cdots, m),$$

where for convenience we let $f_j(x) = F_j^{(0)}(x)$. Hence the set of algebraic inequalities,

$$\sum_{j=1}^{m} c_j F_j^{(k)}(x_1) \geqq 0, \qquad (k = 0, 1, 2, \cdots, m), \tag{4}$$

has a solution for the c's, namely $c_i = \mathbf{c}_i, (i=1,2, \cdots, m)$. The rank of the matrix of the coefficients is m since $D_k(x_1) \neq 0$. Hence the set (4) can not have a non-trivial solution for the case where the equality sign holds throughout, and there therefore exists a solution where the inequality sign holds in at least one instance. It follows by a known theorem* that the associated set of equations,

$$\sum_{k=0}^{m} F_j^{(k)}(x_1) y_{k+1} = 0, \quad (j = 1, 2, \cdots, m), \tag{5}$$

does not have a *definite*† solution. But we have

$$y_1 : y_2 : \cdots : y_{m+1} = D_1(x_1) : D_2(x_1) : \cdots : D_{m+1}(x_1)$$

and they do have a definite solution as the $D_k(x_1)$ are all different from zero and of the same sign. This contradiction proves that the set $\{f_j(x)\}$ has the property (A) and thus completes the proof of the theorem.

The determinants $D_k(x)$ are special cases of the general determinant of order m,

$$F(x) \equiv (F_1^{(k_1)}(x) F_2^{(k_2)}(x) \cdots F_m^{(k_m)}(x)) \equiv \begin{vmatrix} F_1^{(k_1)}(x) & F_1^{(k_2)}(x) & \cdots & F_1^{(k_m)}(x) \\ \cdot & \cdot & \cdot & \cdot \\ F_m^{(k_1)}(x) & F_m^{(k_2)}(x) & \cdots & F_m^{(k_m)}(x) \end{vmatrix},$$

where $k_i \geqq 0$ $(i=1,2, \cdots, m)$.

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* Dines, L. L., *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, vol. 28 (1926), pp. 41–42.

† A solution $(y_1, y_2, \cdots, y_{m+1})$ is said to be a definite solution if each y is different from zero and they are all of the same sign.

THEOREM 2. *If $f_1(x), f_2(x), \cdots, f_m(x)$ possess derivatives of order r, then*

$$\left.\frac{d^rF(x)}{dx^r}\right]_{x=a} = 0, \quad \left[r = 0, 1, 2, \cdots, \sum_{i=1}^{m} k_i + (m(m-1)/2) - 1\right].$$

Since $dF(x)/dx$ may be expressed as a sum of m determinants obtained from $F(x)$ by differentiating the elements of each column in succession we see that when $F(x)$ is written in the notation $(F_1^{(k_1)}(x)F_2^{(k_2)}(x) \cdots F_m^{(k_m)}(x))$, the derivative of $F(x)$ satisfies the same formal rules as the ordinary derivative of a product of m functions. Hence we have

$$(6) \quad \frac{d^rF(x)}{dx^r}$$
$$= \sum_{i_1+i_2+\cdots+i_m=r} \frac{r!}{i_1!i_2!\cdots i_m!}\left(F_1^{(k_1-i_1)}(x)F_2^{(k_2-i_2)}(x)\cdots F_m^{(k_m-i_m)}(x)\right).$$

We here interpret $F_j^{-s}(x)$ $(s>0)$ as $d^sf(x)/dx^s$.

Now $F_j^{(s)}(x)$ vanishes at $x=a$ for $s>0$. Further if any two of the numbers

$$k_1-i_1,\ k_2-i_2,\ \cdots,\ k_m-i_m$$

are equal, the corresponding term in (6) vanishes as two columns of the determinant are identical. Let r' be the value of r for which the sequence

$$k_1-i_1,\ k_2-i_2,\ \cdots,\ k_m-i_m$$

is identical except possibly for order with the sequence

$$0,\ -1,\ -2,\ \cdots,\ -(m-1),$$

that is,

$$r' = \sum_{i=1}^{m} k_1 + m(m-1)/2.$$

Then $d^{r'}F(x)/dx^{r'}$ does not necessarily vanish at $x=a$. If $r<r'$, then $d^rF(x)/dx^r$ does vanish at $x=a$ as we find in each determinant of (6) either a column of zeros or two columns alike. This completes the proof of the theorem. The following corollary results immediately.

COROLLARY. *If*

$$f_1(x),\ f_2(x),\ \cdots,\ f_m(x)$$

are polynomials in x, *then* $D_k(x)$ *has* $x-a$ *as a factor of order* m^2-k+1.

The sufficient condition for the presence of properties (A) and (B) as given by Theorem 1 involves the finding of a single value of x satisfying a certain condition. We give now a test of a different form which involves finding $m+1$ distinct values of x satisfying a given condition.

Let

$$x_1,\ x_2,\ \cdots,\ x_{m+1}$$

be distinct values of x on X and let $(-1)^{k+1}\ B_k$ denote the determinant obtained by striking out the kth column from the matrix:

$$\| f_i(x_j) \|, \qquad (i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, m+1).$$

We have then the following theorem.

THEOREM 3. *If there exist* $m+1$ *distinct values of* x *on* X *such that the determinants* $B_k(k=1,2,\cdots,m+1)$ *are all different from zero and of the same sign, then the set* $\{f_j(x)\}$ *has properties* (A) *and* (B).

This theorem may be proved in a manner similar to the proof of Theorem 1.

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