A NOTE CONCERNING CACTOIDS*

BY J. H. ROBERTS†

A *cactoid*‡ M is a bounded continuous curve lying in space of three dimensions and such that (a) every maximal cyclic curve§ of M is a simple closed surface and (b) no point of M lies in a bounded complementary domain of any subcontinuum of M. There exists a bounded acyclic‖ continuous curve C such that every bounded acyclic continuous curve is homeomorphic with a subset of C. Now Whyburn has shown¶ that with respect to its cyclic elements every continuous curve is acyclic. Moreover the cyclic elements of a cactoid are either points or topological spheres. Thus this question naturally arises: Does there exist a cactoid C such that every cactoid is homeomorphic with a subset of C? The object of the present paper is to answer this question negatively.

THEOREM 1. *There does not exist a cactoid C such that every cactoid is homeomorphic with a subset of C.*

PROOF. Let g be any infinite set of distinct positive integers $d_1, d_2, d_3, \cdots$. Let K denote a non-dense perfect point set on the interval $0 \leqq x \leqq 1$ containing the end points of this interval. The complementary segments of K can be labeled

* Presented to the Society, April 18, 1930.

† National Research Fellow.

‡ See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 81.

§ For a definition of this term, and of the term *cyclic element*, see G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 167.

‖ See T. Wazewski, *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée de Jordan*, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 57. See also Menger, *Über allgemeinen Kurventheorie*, Fundamenta Mathematicae, vol. 10 (1926), p. 108. In his paper *On continua which are disconnected by the omission of any point and some related problems*, Monatshefte für Mathematik und Physik, vol. 35 (1929), p. 136, W. L. Ayres extends this result to unbounded acyclic continuous curves. An *acyclic* continuous curve is one which contains no simple closed curve.

¶ Loc. cit., pp. 167–194.

$s_{ij}(i, j=1, 2, 3, \cdots)$ in such a manner that for every i' and every two distinct points U and V of K there is a j such that the segment $s_{i'j}$ is between U and V. For each i and j there exists a continuum M_{ij} which is the sum of d_i spheres $A_1, A_2, \cdots, A_{d_i}$, where a diameter of $A_k(k \leqq d_i)$ is a subset of the interval s_{ij}, A_k and A_{k+1} are tangent externally, and A_1 and A_{d_i} respectively contain the end points of s_{ij}. Let G_g denote the collection whose elements are the continua $M_{ij}(i, j=1, 2, 3, \cdots)$ and those points of K which do not belong to any continuum M_{ij}. Then G_g is an upper semi-continuous collection, and is an arc with respect to its elements. Moreover, for each i the elements of G_g which are the sum of d_i spheres form a set which is everywhere dense on this arc. Let C_g^* be the point set $K+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}M_{ij}$. Then C_g^* is a cactoid. Let P and Q denote the end points of the interval $0 \leqq x \leqq 1$. Then it is easy to see that if D is any point of C_g^* other that P and Q, there exist in C_g^* arcs PD and QD which have only D in common. Obviously also if g_1 and g_2 are two infinite sets of distinct positive integers and g_1 contains an integer not in g_2 then $C_{g_1}^*$ and $C_{g_2}^*$ are not homeomorphic. Now there exists an uncountable collection (g) such that each element of (g) is an infinite set of distinct positive integers, and for each two elements g_1 and g_2 of (g) there is an integer which belongs to one of them but not to the other.

Suppose C is a cactoid such that every cactoid is homeomorphic with a subset of C. Then for each element g of (g) the set C contains a cactoid C_g which is homeomorphic with the cactoid C_g^* defined above. Let P_g and Q_g be the points of C_g which correspond to the points P and Q under a transformation throwing C_g^* into C_g. As (g) is uncountable it is easy to see that there exists an infinite sequence $g_1, g_2, g_3, \cdots$, of elements of (g) such that P_{g_1} and Q_{g_1}, respectively, are sequential limit points of the sequences $P_{g_2}, P_{g_3}, P_{g_4}, \cdots$, and $Q_{g_2}, Q_{g_3}, Q_{g_4}, \cdots$. As C is a continuous curve and $P_{g_1} \neq Q_{g_1}$, there exists an $n(n>1)$ such that C contains arcs $P_{g_1}P_{g_n}$ and $Q_{g_1}Q_{g_n}$ which have no points in common. Suppose C_{g_n} contains a point D (distinct from P_{g_n} and Q_{g_n}) which does not belong to C_{g_1}. Now C_{g_n} contains arcs $P_{g_n}D$ and $Q_{g_n}D$ having only D in common. Hence there exists an arc XDY in C with only X and Y in C_{g_1}. There exists an arc XBY which is a subset of C_{g_1}. As the maximal cyclic curves of C are spheres it follows that the simple closed curve $XDYBX$ is a subset of a sphere S which belongs to C. Now the arc XBY contains a subarc which is a subset of a sphere T belonging to C_{g_1}. Then S and T have more than one point in common, and hence are identical. Then C_{g_1} contains D, contrary to supposition, whence C_{g_n} is a subset of C_{g_1}. Likewise C_{g_1} is a subset of C_{g_n}. As this is impossible we see that the above supposition has led to a contradiction and the theorem is proved.

In glancing over the proof one can see that the only property used of the topological sphere (which is not also a property of every compact, cyclicly connected continuous curve) is that it is not homeomorphic with a proper subset of itself. Thus the proof suffices for the following theorem.

THEOREM 2. *If M is a class of compact continuous curves whose maximal cyclic curves are homeomorphic but no one is homeomorphic with a proper subset of itself, then there is no universal curve of class M; that is, no curve C of class M such that every curve of class M is homeomorphic with a subset of C.*

THE UNIVERSITY OF PENNSYLVANIA