ABSTRACT DEFINING RELATIONS FOR THE SIMPLE GROUP OF ORDER 5616*

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1. *Generation of G_{5616}.* The collineation group $LF(3, 3)$ of order 5616 leaves the sets of points and lines of the finite geometry $PG(3, 3)$ invariant. The points which are the sets (x_1, x_2, x_3) where x_1, x_2, x_3 are the integers modulo 3 and their corresponding polars with respect to the conic $x_1^2+2x_1x_2+x_2x_3=0$ will be denoted in the following manner.

	Point	Polar		Point	Polar
a	1 0 0	$x_1+x_2=0\theta$	g	1 0 1	$x_1=0\kappa$
b	1 1 1	$x_1+x_3=0\gamma$	h	1 2 0	$x_2+x_3=0\nu$
c	1 1 2	$x_1+x_2+x_3=0\mu$	i	1 2 1	$x_3=0\epsilon$
d	1 2 2	$x_2+2x_3=0\alpha$	j	0 1 1	$2x_1+x_2+x_3=0\delta$
e	1 1 0	$x_1+2x_2+x_3=0\iota$	k	0 0 1	$x_2=0\lambda$
f	0 1 0	$2x_1+x_3=0\zeta$	l	1 0 2	$2x_1+x_2=0\beta$
			m	0 2 1	$x_1+x_2+2x_3=0\eta$

Later we shall have occasion to refer to these polars in connection with the group of isomorphisms.

The collineation group may be generated by two operators of orders three and two respectively which satisfy the necessary conditions

$$S^3 = T^2 = (TS)^{13} = 1.$$

Two operators which belong to the group and possess these properties are

$$S = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}, \quad T = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

For convenience in discussion we shall represent the collineation group as a permutation group on the points of the geometry. The operators S and T then become

$$S = (aib)(cfd)(ehk)(gjm),$$
$$T = (bj)(cm)(ef)(gl),$$
$$R \equiv TS = (aibmfhkedcglj).$$

* Presented to the Society, September 11, 1930.

The group generated by two operators of orders three and two respectively whose product is of the form $6k\pm 1$, where k is an integer, is perfect.* Every perfect group is simple or else it is isomorphic $(\alpha-1)$ to some simple group of composite order. If S and T generate a subgroup of G_{5616}, its order is 432 or less. All the simple groups of orders less than 432 are of orders 60, 168, and 360. Obviously G_{5616} contains no subgroups having an operator of order 13 which is isomorphic to any of the above mentioned simple groups. Since S and T belong to G_{5616} they can generate a group of no higher order. Therefore S and T generate G_{5616}.

The group G_{5616} contains 117 operators of order two† and T is invariant under a G_{48} whose generators are

$$(TR^{-1}TR)^3 = (be)(dh)(fj)(gl),$$
$$(TR^{-3}TR^3)^2 = (bj)(hk)(eg)(fl),$$
$$(TR^{-5}TR^5) = (bc)(di)(gl)(mj).$$

The transformation of S by this G_{48} gives 48 different operators conjugate to S. The group possesses altogether 624 operators conjugate to S‡ contained in 13 sets of 48 conjugates under the G_{48} which leaves invariant. Each set contains one of the following operators:

$S_1 = (aib)(cfd)(ehk)(gjm)$, $TS_1 = (aibmfhkedcglj)$,
$S_2 = S_1^2 = (abi)(cdf)(ekh)(gmj)$,
$S_3 = R^{-1}S_1R = (afl)(bmi)(cgh)(dke)$, $TS_3 = (afdkelhcibjmg)$,
$S_4 = S_3^2 = (alf)(bim)(chg)(dek)$,
$S_5 = R^{-2}S_1R^2 = (bmf)(ced)(glk)(hji)$, $TS_5 = (bihjme)(cfd)(gk)$,
$S_6 = R^{-3}S_1R^3 = (abk)(cgd)(elj)(fhm)$, $TS_6 = (abehmgjk)(cfld)$,
$S_7 = S_6^2 = (akb)(cdg)(ejl)(fmh)$,
$S_8 = R^{-4}S_1R^4 = (adj)(cgl)(eim)(fhk)$, $TS_8 = (adjb)(cehkfimg)$,
$S_9 = S_8^2 = (ajd)(clg)(emi)(fkh)$,
$S_{10} = R^{-6}S_1R^6 = (alj)(bgi)(cmh)(dke)$, $TS_{10} = (alib)(ch)(dkef)(gj)$,

* Brahana, American Journal of Mathematics, vol. 50 (1928), p. 347.
† Miller, Quarterly Journal of Mathematics, vol. 29 (1898), p. 234.
‡ Ibid., p. 234.

$$S_{11}=R^{-10}S_1R^{10}=(aec)(bmf)(hki)(glj),\quad TS_{11}=(aebgjm)(hki)(cf),$$

$$S_{12}\qquad\qquad =(aih)(cke)(dfm)(glj),^{*}\,TS_{12}=(aih)(bgj)(cdf)(emk),$$

$$S_{13}\qquad\qquad =(ajb)(cmk)(eif)(ghl),^{*}\,TS_{13}=(aj)(ck)(fi)(hl).$$

We will denote the sets by the subscripts on the corresponding S's. Members of the pairs (1) and (2), (3) and (4), (6) and (7), and (8) and (9) are distinct since no one of the operators of order two of the G_{48} transforms the corresponding S into its inverse. (1) and (2) are distinct from (3) and (4) since a which is omitted by G_{48} occurs in a cycle of S_1 with i while none of the letters d, h, i, k, which form a transitive constituent of G_{48}, occur in the same cycle with a in S_3. (5) and (11) are distinct since G_{48} omits a. (6) and (8) are distinct since in the product TS a occurs in a cycle of order 8 in one case and in one of order 4 in the other. (10), (12), and (13) are distinct since the products TS are all of different orders. None of the operators $S_5, S_6, \cdots, S_{13}$ generate the group with T because the resulting groups are intransitive on 13 letters. It follows that there are two sets of operators and their inverses which generate the group with T.

THEOREM 1. *If an operator of order two and one of order three generate G_{5616}, their product is of order thirteen.*

The theorem follows since an operator of order three containing three cycles cannot generate the group with T because T can have only four cycles and the resulting group could not be transitive on 13 letters.

From §1 we obtained the following correlation of the plane:

$$(a\theta)(b\gamma)(c\mu)(d\alpha)(e\iota)(f\zeta)(g\kappa)(h\nu)(i\epsilon)(j\delta)(k\lambda)(l\beta)(m\eta) = P.$$

This is an outer isomorphism which leaves T invariant,

$$S' = (aib)(cfd)(ehk)(gjm),$$

$$T' = (bj)(cm)(ef)(gl).$$

The operator $S'=(aib)(cfd)(ehk)(gjm)$ written on points performs the permutation $\Sigma=(\alpha\nu\zeta)(\beta\epsilon\theta)(\delta\eta\kappa)(\iota\mu\lambda)$ written on lines. For a representation of S' on symbols for lines we may use Σ. The correlation P transforms points into lines,

$$P\ \Sigma\ P = (ali)(cke)(dhf)(gjm).$$

§ Operators conjugate to S_{12} and S_{13} are in the group $\{T, S_{10}\}$.

Also there exists an operator of the G_{48} which transforms $P\Sigma P$ into S^2. Such a one is $(bl)(ef)(dk)(gj)$. Since P which is an outer isomorphism transforms the operators of the first set into their inverses, every other outer isomorphism which leaves T invariant has the same property. Hence there is no outer isomorphism which leaves T invariant and at the same time transforms the first set into the second and it follows that the two sets of generators satisfy different defining relations.

2. *The Cole Group of Order* 432. Cole has investigated the groups of degree nine and found that there is a single transitive substitution group of order 432 representable on nine letters.*

The group with which we have to deal is written intransitively on thirteen letters and contains two transitive constituents one on four letters and one on nine letters. Hence it follows that the latter group is equivalent to the one described by Cole. We have a priori that the foregoing is a maximal subgroup of G_{5616}. In order to define this group abstractly it will be convenient first to determine a set of defining relations for the maximal subgroup H, of order 432.

THEOREM 2. *Two operators of orders three and two respectively which satisfy the following conditions,*

$$S^3 = T^2 = (ST)^8 = 1, \tag{1}$$

$$TS^2TS(ST)^4S^2TST = S^2TS(ST)^4S^2TS, \tag{2}$$

$$(S^2TST)^6 = 1, \tag{3}$$

generate the abstract group of order 432 *which is doubly transitive on nine letters, or a group of lower order to which* G_{432} *is isomorphic.*

If S and T are of orders three and two respectively and $(ST)^4$ is identity, we get the octahedral group; if $(ST)^4$ is invariant and not identity, we get the group of order 48.

By virtue of (1), $(ST)^4$ is of order two; and as consequences of (2) and (3), it will be proved that there exists a set of nine conjugate operators arising from the transformation of $(ST)^4$ by T and powers of S. Denoting $(ST)^4$ by T_1, let us consider the following transforms:

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* Bulletin of the New York Mathematical Society, vol. 2 (1892-93), pp. 254-256.

$$
\begin{aligned}
&T_1, & STST_1S^2TS^2 &= T_6,\\
&S^2T_1S = T_2, & TSTST_1S^2TS^2T &= T_7,\\
&ST_1S^2 = T_3, & S^2TSTST_1S^2TS^2TS &= T_8,\\
&TST_1S^2T = T_4, & STSTST_1S^2TS^2TS^2 &= T_9.\\
&S^2TST_1S^2TS = T_5, & &
\end{aligned}
$$

Obviously in order that this conjugate set contain nine and only nine operators it will be sufficient to show that T_8 and T_9 are permutable with T, that is, $TT_8T=T_8$, and $TT_9T=T_9$. The foregoing results directly from conditions (2) and (3). Writing (2) in the following form and remembering that $S^2(ST)^4S=T(ST)^4T$, we have

$$
\begin{aligned}
&S^2TSTS^2TS(ST)^4S^2TSTS^2TS=(ST)^4,\\
&TS^2TSTS^2TS(ST)^4S^2TSTS^2TST = T(ST)^4T = S^2(ST)^4S,\\
&STS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2 = (ST)^4,\\
\text{(i)}\quad &TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2T = T(ST)^4T,\\
&S^2TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2TS=S^2T(ST)^4TS=S(ST)^4,\\
&S^2TS^2TSTS^2TSTS^2TS(ST)^4S^2TSTS^2TSTS^2TST=TS(ST)^4S^2T.
\end{aligned}
$$

It follows from condition (3) that $T(S^2TST)^3T=(S^2TST)^3$ and the last expression becomes

$$S^2TSTS^2TSTS^2TST(ST)^4TS^2TSTS^2TSTS^2TS = TS(ST)^4S^2T.$$

Continuing in the same manner this expression reduces to

$$STS^2TSTS^2T(ST)^4TSTS^2TSTS^2 = TSTS(ST)^4S^2TS^2T.$$

Transforming by S, we have, after replacing $T(ST)^4T$ by $S^2(ST)^4S$,

$$\text{(ii)}\qquad S^2TSTS(ST)^4S^2TS^2TS = TS^2TSTS(ST)^4S^2TS^2TST,$$

which gives the desired relation between T and T_8. On transforming both members of equation (ii) above first by S and then T we have

$$
\begin{aligned}
TSTSTS(ST)^4S^2TS^2TS^2T &= TS^2TS^2TSTS(ST)^4S^2TS^2TSTST \\
&= TS^2TS^2TS^2TTS^2TS(ST)^4S^2TSTTSTSTST \\
\text{(iii)} \qquad &= (ST)^3S^2(ST)^4STS^2(TS^2)^2 \\
&= STSTSTT(ST)^4S^2TS^2TS^2 \\
&= STSTS(ST)^4S^2TS^2TS^2,
\end{aligned}
$$

which expresses the necessary relation between T and T_9. By the aid of (iii) we can show that T_1T_2 is of order three. Since $TT_9T=T_9$, we may write

$$
\begin{aligned}
&STSTSTS(ST)^4S^2TS^2TS^2TS^2 = ST_9S^2 = T_8, \\
&ST_9S^2 = (ST)^4(TS)^4(ST)^4 = T_8 = TT_8T, \\
&(ST)^4(TS)^4(ST)^4 = (TS)^4(ST)^4(TS)^4, \\
&T_1T_2 = (ST)^4S^2(ST)^4S = (ST)^4(TS)^4, \\
&(T_1T_2)^3 = (ST)^4(TS)^4(ST)^4(TS)^4(ST)^4(TS)^4 = T_8^2 = 1.
\end{aligned}
$$

Since T_1T_2 is of order three and T_1T_2 is transformed into T_3T_1 by S^2 the product T_1T_3 is necessarily of order three.

The group generated by T_1T_2 and T_1T_3 is abelian.* From the third of the defining relations given in the first paragraph of this section we have

$$(S^2TST)^6 = 1,$$

which implies

$$(S^2TS^2TS^2)^6 = 1,$$

and

$$S^2TS^2TS^2 = S^2(TS^2)^2 = T(ST)^5,$$

$$
\begin{aligned}
&T(ST)^5T(ST)^5T(ST)^5T(ST)^5T(ST)^5T(ST)^5 = 1, \\
&T(ST)^4S(ST)^4S(ST)^4S(ST)^4S(ST)^4S(ST)^4ST = 1, \\
&(ST)^4S(ST)^4S(ST)^4S = S^2(ST)^4S^2(ST)^4S^2(ST)^4, \\
&S^2(ST)^4S(ST)^4S(ST)^4S^2 = S(ST)^4S^2(ST)^4S^2(ST)^4S,
\end{aligned}
$$

$$\text{(iv)} \quad (ST)^4S^2(ST)^4S(ST)^4S(ST)^4S^2 = (ST)^4S(ST)^4S^2(ST)^4S^2(ST)^4S.$$

The equations (iv) express the desired relation between T_1T_2 and T_1T_3, namely:

$$\text{(v)} \qquad T_1T_2T_1T_3 = T_1T_3T_1T_2.$$

Furthermore the products T_1T_2 and T_1T_3 are different and not

* Miller, Quarterly Journal of Mathematics, vol. 33 (1901), pp. 76–79.

inverses of each other. If $T_1T_2 = T_1T_3$, $T_2 = T_3$ and this is impossible when T_1 is not invariant. Also, if $T_1T_2 = T_3T_1$, from (v) $T_3^2 = (T_1T_2)^2 = 1$, which is impossible when T_1 is not invariant since (T_1T_2) is then of order three. Hence the group $\{T_1T_2, T_1T_3\}$ is the abelian group of order nine and type (1, 1).

What has been proved so far is sufficient to state that the group generated by S and T satisfying the given conditions contains a generalized dihedral group of order 18 as an invariant subgroup since it is $\{G_9, T_1\}$ and T_1 transforms T_1T_2 and T_1T_3 into their inverses. We have shown that $(ST)^4$ is in an invariant subgroup I_{18}. Since I_{18} does not contain S or T because S is not permutable with T_1T_2 and T does not transform T_1T_3 into T_3T_1, it follows that the quotient group corresponding to I_{18} is generated by operators of orders three and two respectively, whose product is of order four, since $(ST)^4$ is in I_{18}. Hence the quotient group is the symmetric group of order 24, and the order of H is 432.

It might be observed that the group is representable on the nine congugate T's which we have discussed. Since S and T permute the T's according to the following permutations on the subscripts,

$$S = (123)(456)(789),$$
$$T = (12)(34)(67),$$

which satisfy relations (1), (2), and (3) it follows that the group is representable on nine letters and this identifies it with Cole's group. It is now clear that the structure of the entire group coincides with the description given by him, and hence it is doubly transitive.

3. *Definition of the Whole Group* G_{5616}. In §1 we discussed some of the abstract properties of the generators of the group and found that G_{5616} possesses two sets of generators which satisfy different defining relations. In order to define the whole group we shall employ the results of §2 since the largest subgroup H of order 432 forms the basis for writing all the operators of G_{5616} on co-sets. Every operator of G_{5616} occurs in one of the following co-sets,

$$\text{(i)} \qquad H, RH, R^2H, \cdots, R^{12}H.$$

Since the result of multiplication of any h_i belonging to H on the right by any power of R is equivalent to that of multiplying another h_j on the left by some power of R it follows that integers β_i exist such that

$$\text{(ii)}\qquad \begin{gathered} R^{\alpha}\bar{S}R^{\beta_1} = h_i, \quad R^{\alpha}\bar{S}^2R^{\beta_2} = h_j, \quad R^{\alpha}TR^{\beta_3} = h_k, \\ R^{-9}SR^{9} = \bar{S}, \text{ and } H = \{\bar{S}, T\}, \end{gathered}$$

are true for all values of α from 1 to 12 inclusive.

On the basis of

$$\text{(iii)}\qquad R^2\bar{S}R^{10} = h_1, \quad R^3\bar{S}R^6 = h_2, \quad R^4\bar{S}R^8 = h_3,$$

where h_1, h_2, h_3 are operators of H, we shall show that the β_i's of equations (ii) are uniquely determined and that h_i, h_j, h_k are expressible in terms of the generators of H. Writing $R^{\alpha}\bar{S}^2R^{\beta}$ in the form $R^{\alpha-1}R^{-9}TSS^2TSR^9R^{\beta-1}$, we see that it is equal to $R^{\alpha-1}\bar{S}R^{\beta-1}$. We can treat $R^{\alpha}TR^{\beta}$ in the same manner. Since $R^{\alpha}TR^{\beta} = R^{\alpha}TTSR^{\beta-1} = R^{\alpha}SR^{\beta-1} = R^{\alpha+9}R^{-9}SR^9R^{\beta-10}$, we have $R^{\alpha}TR^{\beta} = R^{\alpha+9}\bar{S}R^{\beta-10}$. These two statements are sufficient to express the equations (ii) in terms of h_1, h_2, h_3, or, what is the same, in terms of the generators of H. We proceed as follows:

$$h_1h_2 = R^2\bar{S}R^{10}R^3\bar{S}R^6 = R^2\bar{S}^2R^6 = R\bar{S}R^5,$$

where h_1, h_2, h_3 are given by (iii). We have also

$$\begin{aligned}
h_2^{-1}h_3 &= R^7\bar{S}^2R^{10}R^4\bar{S}R^8 = R^6\bar{S}R^9R^4\bar{S}R^8 = R^6\bar{S}^2R^8 = R^5\bar{S}R^7,\\
h_2^{-1}h_3^{2} &= R^6\bar{S}^2R^8R^5\bar{S}^2R^9 = R^6\bar{S}R^9,\\
(h_1h_2)^{-1} &= R^8\bar{S}^2R^{12} = R^7\bar{S}R^{11},\\
(h_1h_2)^{-1}\bar{S} &= R^8\bar{S}^2R^{12}R\bar{S}^2R = R^8\bar{S}R,\\
T &= SS^2T = R^9R^{-9}SR^9R^3 = R^9\bar{S}R^3,\\
[(h_1h_2)^{-1}\bar{S}]^{-1}T &= R^{11}\bar{S}R^4R^9\bar{S}R^3 = R^{11}\bar{S}^2R^3 = R^{10}\bar{S}R^2,\\
[(h_1h_2)^{-1}\bar{S}]^{-1} &= R^{12}\bar{S}^2R^5 = R^{11}\bar{S}R^4, \quad \bar{S}^2 = (R\bar{S}^2R)^{-1} = R^{12}\bar{S}R^{12}.
\end{aligned}$$

Since $R^{\alpha}\bar{S}R^{\beta} = R^{\alpha+1}\bar{S}^2R^{\beta+1}$, we have for the expressions $R^{\alpha}\bar{S}^2R^{\beta_i}$ where $\alpha = (1, 2, 3, \cdots, 12)$:

$R\bar{S}^2R$	$R^4\bar{S}^2R^7$	$R^7\bar{S}^2R^{10}$	$R^{10}\bar{S}^2R^4$
$R^2\bar{S}^2R^6$	$R^5\bar{S}^2R^9$	$R^8\bar{S}^2R^{12}$	$R^{11}\bar{S}^2R^3$
$R^3\bar{S}^2R^{11}$	$R^6\bar{S}^2R^8$	$R^9\bar{S}^2R^2$	$R^{12}\bar{S}^2R^5$

Also for $R^{\alpha+9}\bar{S}R^{\beta-10}$, we have $R^{\alpha}TR^{\beta}$, where α ranges over the same values as above:

$$\begin{aligned}
RTR^{12} &= R^{10}\bar{S}R^{2}, & R^{5}TR^{2} &= R\bar{S}R^{5}, & R^{9}TR^{4} &= R^{5}\bar{S}R^{7},\\
R^{2}TR &= R^{11}\bar{S}R^{4}, & R^{6}TR^{7} &= R^{2}\bar{S}R^{10}, & R^{10}TR^{6} &= R^{6}\bar{S}R^{9},\\
R^{3}TR^{9} = R^{12}\bar{S}R^{12} &= \bar{S}^{2}, & R^{7}TR^{3} &= R^{3}\bar{S}R^{6}, & R^{11}TR^{8} &= R^{7}\bar{S}R^{11},\\
R^{4}TR^{10} &= \bar{S}, & R^{8}TR^{5} &= R^{4}\bar{S}R^{8}, & R^{12}TR^{10} &= R^{8}\bar{S}R\,.
\end{aligned}$$

Hence all of the equations (ii) have solutions in terms of h_1, h_2, h_3, and the β_i's are uniquely determined. In a similar manner when $\bar{S}=R^{-1}SR$ which gives another set of conditions we can also show that the equations are reducible to expressions in terms of the generators of H and that the β_i's are uniquely determined.

Having shown that we can establish sufficient conditions which enable us to specify the order of G, we may now state the following theorem.

THEOREM 3. *Two operators T and S of orders two and three with product of order thirteen generate G_{5616} if and only if they satisfy the further relations*:

$$\bar{S}^{3} = T^{2} = (\bar{S}T)^{8} = 1, \tag{1}$$

$$T\bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}T = \bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}, \tag{2}$$

$$(\bar{S}^{2}T\bar{S}T)^{6} = 1, \tag{3}$$

where $\bar{S}=R^{-9}SR$ *or* $R^{-1}SR$ *and* $R\equiv TS$, *and either* (4), (5), (6), *or* (4′), (5′), (6′),

$$R^{2}\bar{S}R^{10} = T\bar{S}^{2}T\bar{S}T, \tag{4}$$

$$R^{3}\bar{S}R^{6} = T\bar{S}(\bar{S}T)^{4}\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}^{2}, \tag{5}$$

$$R^{4}\bar{S}R^{8} = T\bar{S}^{2}T\bar{S}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T\bar{S}^{2}T\bar{S}, \tag{6}$$

where $\bar{S}=R^{-9}SR$;

$$R^{3}\bar{S}R^{9} = T\bar{S}^{2}T\bar{S}T, \tag{4′}$$

$$R^{5}\bar{S}R^{4} = T\bar{S}^{2}T\bar{S}^{2}T\bar{S}(\bar{S}T)^{4}\bar{S}^{2}T, \tag{5′}$$

$$R^{7}\bar{S}R^{5} = \bar{S}(\bar{S}T)^{4}\bar{S}(\bar{S}T)^{3}\bar{S}, \tag{6′}$$

where $\bar{S}=R^{-1}SR$.

If two pairs of operators ρ and τ and Σ and τ which generate the group are so selected that one of them ρ and τ satisfy a particular one of the sets of conditions imposed on S and T and if ρ happens to be the inverse of Σ then Σ and τ satisfy identically the same relations as ρ and τ. This follows from the fact that there is an outer isomorphism which transforms ρ into its inverse and leaves T invariant.

THE UNIVERSITY OF ILLINOIS

ONE-PARAMETER LINEAR FUNCTIONAL GROUPS IN SEVERAL FUNCTIONS OF TWO VARIABLES*

BY A. D. MICHAL

Let

$$
{}^{j}_{i}H(x, y), \qquad (i, j = 1, 2, \cdots, n), \tag{1}
$$

be a given set of n^2 real continuous functions of the two real variables x and y defined over the triangular region $T: a \leqq x \leqq y \leqq b$. Consider the system of *n integro-differential equations*†

$$
\frac{\partial_i z(x, y; \tau)}{\partial \tau} = {}^{j}_{i}H \star {}_{j}z(x, y; \tau), \quad (i = 1, 2, \cdots, n), \tag{2}
$$

in the n unknown functions ${}_1z(x, y; \tau)$, ${}_2z(x, y; \tau)$, $\cdots$, ${}_nz(x, y; \tau)$. The symbol $\star$ in (2) stands for *Volterra's operation composition of the first kind*

$$
{}^{j}_{i}H \star {}_{j}z(x, y; \tau) = \int_x^y {}^{j}_{i}H(x, s)\, {}_{j}z(s, y; \tau) ds. \tag{3}
$$

We assume that the reader is conversant with the theory of permutable functions and functions of composition, a subject initiated by the illustrious Vito Volterra.‡ Griffith C. Evans§

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* Presented to the Society, December 30, 1930.

† Throughout the whole paper we shall adhere to the convention of letting a repetition of an index in a term once as a subscript and once as a superscript denote summation with respect to that index over the values $1, 2, \cdots, n$.

‡ See, for example, Volterra et Pérès, *Leçons sur la Composition et les Fonctions Permutables*.

§ See his Cambridge Colloquium Lectures, *Functionals and their Applications.*