If two pairs of operators ρ and τ and Σ and τ which generate the group are so selected that one of them ρ and τ satisfy a particular one of the sets of conditions imposed on S and T and if ρ happens to be the inverse of Σ then Σ and τ satisfy identically the same relations as ρ and τ. This follows from the fact that there is an outer isomorphism which transforms ρ into its inverse and leaves T invariant.

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ONE-PARAMETER LINEAR FUNCTIONAL GROUPS IN SEVERAL FUNCTIONS OF TWO VARIABLES*

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Let

$$ {}^{j}_{i}H(x, y), \qquad (i, j = 1, 2, \cdots, n), \tag{1} $$

be a given set of n^2 real continuous functions of the two real variables x and y defined over the triangular region $T: a \leqq x \leqq y \leqq b$. Consider the system of *n integro-differential equations*†

$$ \frac{\partial {}_{i}z(x, y; \tau)}{\partial \tau} = {}^{j}_{i}H \star {}_{j}z(x, y; \tau), \quad (i = 1, 2, \cdots, n), \tag{2} $$

in the n unknown functions ${}_{1}z(x, y; \tau)$, ${}_{2}z(x, y; \tau)$, $\cdots$, ${}_{n}z(x, y; \tau)$. The symbol $\star$ in (2) stands for *Volterra's operation composition of the first kind*

$$ {}^{j}_{i}H \star {}_{j}z(x, y; \tau) = \int_{x}^{y} {}^{j}_{i}H(x, s)\, {}_{j}z(s, y; \tau) ds. \tag{3} $$

We assume that the reader is conversant with the theory of permutable functions and functions of composition, a subject initiated by the illustrious Vito Volterra.‡ Griffith C. Evans§

* Presented to the Society, December 30, 1930.

† Throughout the whole paper we shall adhere to the convention of letting a repetition of an index in a term once as a subscript and once as a superscript denote summation with respect to that index over the values $1, 2, \cdots, n$.

‡ See, for example, Volterra et Pérès, *Leçons sur la Composition et les Fonctions Permutables.*

§ See his Cambridge Colloquium Lectures, *Functionals and their Applications.*

has given some very noteworthy contributions to this subject.

Define a set of n^2 functions ${}^{j}_{i}K(x, y; \tau)$ in terms of the given set of functions ${}^{j}_{i}H(x, y)$ by

$$(4) \qquad {}^{j}_{i}K(x, y; \tau) = {}^{j}_{i}H(x, y)\tau + {}^{k}_{i}H\star{}^{j}_{k}H(x, y)\frac{\tau^2}{2!} + {}^{k}_{i}H\star{}^{l}_{k}H\star{}^{j}_{l}H(x, y)\frac{\tau^3}{3!} + \cdots .$$

For $n = 1$, this set reduces to the well known Volterra transcendental.

THEOREM 1. *Under the above hypotheses on the given functions* ${}^{j}_{i}H(x, y)$, *the system of integro-differential equations* (2) *possesses one and only one solution* ${}_{1}z(x, y; \tau), \cdots, {}_{n}z(x, y; \tau)$ *analytic in* τ (*in fact entire in* τ), *continuous in x and y in the region T and such that it takes on the initial conditions*

$$(5) \qquad {}_{i}z(x, y; 0) = {}_{i}z(x, y),$$

(${}_{i}z(x, y)$ *continuous in* T) *for* $\tau = 0$. *This unique solution can be written in the form*

$$(6) \qquad {}_{i}z(x, y; \tau) = {}_{i}z(x, y) + {}^{j}_{i}K(\cdots; \tau)\star{}_{j}z(x, y),$$

where the n^2 *functions* ${}^{j}_{i}K$ *are the generalized Volterra transcendentals* (4).

The proof of this theorem is obtained by throwing the integro-differential system (2) into the form of a system of non-linear integral equations

$$(7) \qquad {}_{i}z(x, y; \tau) = {}_{i}z(x, y) + \int_0^\tau {}^{j}_{i}H\star{}_{j}z(x, y; \sigma)d\sigma,$$

and then solving this system by the method of successive approximations. We are thus led to consider the functions ${}_{i}z(x, y; \tau)$ given by (6). By calculation one sees that this set of functions is a formal solution of the integro-differential system and takes on the given initial conditions. In fact the transcendentals ${}^{j}_{i}K(x, y; \tau)$ satisfy the integro-differential equations

$$(8) \qquad \frac{\partial\,{}^{j}_{i}K(x, y; \tau)}{\partial \tau} = {}^{j}_{i}H(x, y) + {}^{k}_{i}H\star{}^{j}_{k}K(x, y; \tau),$$

and hence the ${}_iz(x, y; \tau)$ of (6) yield a formal solution of the system (2).

It follows directly from the definition (4) that the generalized Volterra transcendentals are *entire* functions of τ and continuous in x and y throughout the region T. Hence the set of functions (6) form an actual solution of (2) of the required sort.

If ${}_iw(x, y; \tau)$ is another solution of (2) of the required sort, it follows from (7) that it is possible to find a positive number $A<1$ such that

$$\max\left| {}_iz(x, y;\tau) - {}_iw(x, y;\tau) \right| \leqq A \max\left| {}_iz(x, y;\tau) - {}_iw(x, y;\tau) \right|,$$

for sufficiently small values of τ. But ${}_iz$ and ${}_iw$ are analytic functions of τ. Hence

$${}_iz(x, y; \tau) \equiv {}_iw(x, y; \tau).$$

This shows that the solution of (2) of the required sort is unique, which completes the proof of our theorem.

Thus the integro-differential system (2) generates a continuous one-parameter family of functional transformations (6). Moreover, one can verify that the generalized Volterra transcendentals ${}^j_iK$ possess the following integral addition theorem:

$$\text{(9)} \quad {}^j_iK(x, y; \tau_1 + \tau_2) = {}^j_iK(x, y; \tau_1) + {}^j_iK(x, y; \tau_2) + {}^j_kK(\cdots; \tau_2) \star {}^k_iK(x, y; \tau_1),$$

and that this integral addition theorem characterizes the n^2 transcendentals ${}^j_iK(x, y; \tau)$. We have thus arrived at the following fundamental theorem.

THEOREM 2. *The integro-differential equations* (2) (*infinitesimal transformations*) *generate the one-parameter continuous group of linear functional transformations* (6) *in the functions* ${}_1z(x, y)$, ${}_2z(x, y), \cdots, {}_nz(x, y)$. *Moreover the identity transformation is obtained by putting the parameter* τ *equal to zero.*

Consider the particular case in which

$$\text{(10)} \quad \begin{cases} {}^j_iH(x, y) = 1, & \text{if } j = n - i + 1, \\ \phantom{{}^j_iH(x, y)} = 0, & \text{otherwise}. \end{cases}$$

The integro-differential system (2) for this case becomes

$$\frac{\partial_i z(x, y; \tau)}{\partial \tau} = 1\star_{n-i+1} z(x, y; \tau). \tag{11}$$

By calculation we obtain

$$\left\{\begin{aligned} {}_i^jK(x, y; \tau) &= \sum_{n=1}^{\infty} \frac{\tau^{2n}(y - x)^{2n-1}}{(2n)!(2n - 1)!}, \text{ if } j = i, \\ &= \sum_{n=1}^{\infty} \frac{\tau^{2n-1}(y - x)^{2n-2}}{(2n - 1)!(2n - 2)!}, \text{ if } j = n - i + 1, \\ &= 0, \text{ otherwise.} \end{aligned}\right. \tag{12}$$

These expressions can be written in terms of the Bessel function $J_1(z)$ of order one.* Putting $u = y - x$, we obtain

$$\left\{\begin{aligned} {}_i^jK(x, y; \tau) &= \frac{\tau^{1/2}u^{-1/2}}{2}\{I_1[2(\tau u)^{1/2}] + J_1[-2(\tau u)^{1/2}]\}, \text{if } j = i, \\ &= \frac{\tau^{1/2}u^{-1/2}}{2}\{I_1[2(\tau u)^{1/2}] - J_1[-2(\tau u)^{1/2}]\}, \\ &\qquad\qquad \text{if } j = n - i + 1, \\ &= 0, \text{ otherwise.} \end{aligned}\right. \tag{13}$$

In these formulas,

$$I_1(z) = -iJ_1(iz), \quad (i = (-1)^{1/2}).$$

For $j = i$ the integral addition theorems (9) for the special case (13) yield the addition theorem

$$\left\{\begin{aligned} &(\tau_1 + \tau_2)^{1/2}u^{-1/2}A[(\tau_1 + \tau_2)u] \\ &\quad = \tau_1^{1/2}u^{-1/2}A[\tau_1 u] + \tau_2^{1/2}u^{-1/2}A[\tau_2 u] \\ &\qquad + (\tau_1\tau_2)^{1/2}\int_0^u [(u - v)v]^{-1/2} \\ &\qquad \cdot \{J_1[-2(\tau_2(u - v))^{1/2}]J_1[-2(\tau_1 v)^{1/2}] \\ &\qquad + I_1[2(\tau_2(u - v))^{1/2}]I_1[2(\tau_1 v)^{1/2}]\}\,dv, \end{aligned}\right. \tag{14}$$

where

$$A[\tau u] = J_1[-2(\tau u)^{1/2}] + I_1[2(\tau u)^{1/2}].$$

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* A. D. Michal, Annals of Mathematics, vol. 26 (1925), for the corresponding case of the Volterra transcendental.

By an application of a known theorem* one can verify that $I_1(z)$ possesses the following integral addition theorem:

$$(15)\quad \begin{cases} (\tau_1 + \tau_2)^{1/2}u^{-1/2}I_1[2((\tau_1 + \tau_2)u)^{1/2}] \\ = \tau_1^{1/2}u^{-1/2}I_1[2(\tau_1 u)^{1/2}] + \tau_2^{1/2}u^{-1/2}I_1[2(\tau_2 u)^{1/2}] \\ + (\tau_1\tau_2)^{1/2}\displaystyle\int_0^u [(u - v)v]^{-1/2}I_1[2(\tau_2(u - v))^{1/2}]I_1[2(\tau_1 v)^{1/2}]dv. \end{cases}$$

Moreover

$$J_1(z) = iI_1(- iz), \qquad (i = (- 1)^{1/2}).$$

Hence by calculation we see that $J_1[-2(\tau u)^{1/2}]$ has precisely the same integral addition theorem as that of $I_1[2(\tau u)^{1/2}]$.

The remaining relations (9) for the particular system (10) do not yield any essentially new integral addition theorems for the Bessel functions.

THEOREM 3. *The group properties of the continuous one-parameter family of functional transformations that is generated by the integro-differential system* (11) *are translated by the integral addition theorem*

$$\begin{aligned} &(\tau_1 + \tau_2)^{1/2}u^{-1/2}J_1[- 2((\tau_1 + \tau_2)u)^{1/2}] \\ &= \tau_1^{1/2}u^{-1/2}J_1[- 2(\tau_1 u)^{1/2}] + \tau_2^{1/2}u^{-1/2}J_1[- 2(\tau_2 u)^{1/2}] \\ &+ (\tau_1\tau_2)^{1/2}\int_0^u [(u - v)v]^{-1/2}J_1[- 2(\tau_2(u - v))^{1/2}]J_1[- 2(\tau_1 v)^{1/2}]dv \end{aligned}$$

for the Bessel function $J_1(z)$ of order one.

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* H. P. Thielman, Annals of Mathematics, vol. 31 (1930), p. 193.