EXTENSION OF A THEOREM OF MAZURKIEWICZ*

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S. Mazurkiewicz,† in answer to a question proposed by B. Knaster,‡ has shown that if A is a closed point set in E_n (euclidean space of n dimensions) which is homeomorphic with a subset of E_{n-1}, then all points of A are accessible from the complementary set, $E_n - A$. The question naturally arises, then, as to whether the points of A are *regularly* § accessible from $E_n - A$. It will be shown in the present paper that this is indeed the case.

We shall precede our proof by two theorems which, we believe, are themselves of fundamental importance. Following Mazurkiewicz' notation, we shall denote by $S_n(p, \rho)$ a spherical neighborhood of a point p of E_n with radius ρ; by $\phi(A)$, the subset of E_{n-1} that is homeomorphic with A; and if X is any subset of A, by $\phi(X)$ we denote that subset of $\phi(A)$ that corresponds to X under the homeomorphism between A and $\phi(A)$. Also, following the usual custom, if M is a point set, by $\overline{M}$ we shall denote the set M together with all of its limit points.

Evidently the proof given by Mazurkiewicz for his *Lemme* establishes the following more general lemma.

LEMMA 1. *Let P be a point of A, D a domain¶ containing P, and G a component of $D - A \cdot D$ such that $\overline{G} \supset P$. Then, if D_1 is a bounded domain such that $D_1 \subset D$ and $D_1 \supset P$, there is a component G_1 of $G \cdot D_1$ such that $P \subset \overline{G}_1$.*

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† *Sur un problème de M. Knaster*, Fundamenta Mathematicae, vol. 13 (1929), pp. 146–150.

‡ See Fundamenta Mathematicae, vol. 8 (1926), Problem 43, p. 376.

§ A point P of a point set M is said to be *regularly* accessible from a point set R of which P is a limit point provided that for every $\epsilon > 0$ there exists a positive number δ such that if Q is a point of R whose distance from P is less than δ, then there is an arc from P to Q whose diameter is less than ϵ and which lies, except for P, wholly in R. See G. T. Whyburn, this Bulletin, vol. 34 (1928), p. 509.

¶ By *domain* we mean a connected open subset of the space under consider ation. The domain D may, of course, be E_n, in which case the component G of this lemma will necessarily exist, due to the invariance of dimensionality under analysis situs transformations.

THEOREM 1. *If D is a bounded domain of E_n, and a component C of $A \cdot D$ separates* D, then the set $\phi(C)$ is a domain of E_{n-1} whose boundary is $\phi(\overline{C}-C)$.*

PROOF. Since, due to similarities between the proof and that given by Mazurkiewicz for his *Lemme.* we can conserve space by referring to the latter, we shall endeavor to retain most of his notation in this connection.

We can assume that $A=\overline{C}$; then the set $\phi(A)$ does not fill up E_{n-1}. Hence $E_n - A$ is connected,† and if a is a point of C there exists, by the above Lemma, a component G of the set

$$(1) \qquad (E_n - C)\cdot D$$

such that $\overline{G} \supset a$. Let $B = A \cdot F = \overline{C} - C$,‡ where F is the boundary of D, and let H be the component of $E_{n-1} - \phi(B)$ that contains $\phi(a)$. We shall show that $H = \phi(C)$.

Since C separates D, there are two points, c and c_1, in (1), which do not lie in the same component of (1); we may suppose that $G \supset c$. Then $\overline{G} - G$ is a cut of E_n between c and c_1, and accordingly contains an *irreducible cut*, L, of E_n, between c and c_1. Let

$$(2) \qquad L = L_1 + L_2,$$

where

$$(2') \qquad L_1 = L \cdot A, \qquad L_2 = L \cdot F.$$

It is clear that $L_1 \cdot D \neq 0$, and hence $L \cdot C \neq 0$, all points of A in D belonging to C. We now note that

$$(3) \qquad \phi(C) \subset H,$$

since $\phi(C)\cdot\phi(B)=0$ and $\phi(C)\supset\phi(a)$. Also, that

$$(4) \qquad \phi(L_1)\cdot H \subset \phi(C),$$

since

$$\phi(L_1)\cdot H \subset \phi(A)\cdot H \subset [\phi(B)+\phi(C)]\cdot H = \phi(C)\cdot H = \phi(C).$$

* That is, there exist, in $D-C$, two points P and Q which are not joined by any subcontinuum of $D-C$.

† See P. Alexandroff, *Sur la décomposition de l'espace par les ensembles fermés*, Comptes Rendus, vol. 184 (1927), pp. 425–428.

‡ That $B \neq 0$ is an immediate consequence of the Alexander Addition Theorem. (See proof of Theorem 2 below.)

Suppose that H contains a point which is not in $\phi(L_1)$. Then we can express H as the sum of two mutually exclusive sets, H_1 and H_2, where

$$(5) \qquad H_1 = H \cdot \phi(L_1), \quad H_2 = H - H_1.$$

Since $\phi(L_1)$ is closed, it follows from (5) that H_1 is closed in H. Then, since H is connected, H_1 contains a limit point, $\phi(P)$, of H_2.

Let ϵ be a positive number less than $\rho[\phi(P), \phi(B)]$, and such that $F_{n-1}[\phi(P), \epsilon]$ contains a point, Q, of H, that is not in $\phi(L_1)$. Such a point exists, of course, since P is a limit point of the set of such points. Let

$$(6) \qquad \phi(L_1) \cdot \bar{S}_{n-1}[\phi(P), \epsilon] = \phi(L_3),$$

$$(7) \qquad \phi(L_1) \cdot \{E_{n-1} - S_{n-1}[\phi(P), \epsilon]\} = \phi(L_4).$$

That $\phi(L_3) \neq 0$ is obvious, since $\phi(P) \subset \phi(L_3)$. That $\phi(L_4) \neq 0$ follows from the following considerations. If we denote the set of points of L that are not in L_2 by L', then L_2 contains a limit point, x, of L'. Since $L' \subset C$, the point x is contained in $\bar{C} - C = B$. By a theorem of Miss Mullikin,* the continuum L contains a connected set, L'', which contains P and has at least one limit point in B, but contains no point of B. It is easy to see that $L'' \subset L_1$, and consequently that $\phi(L'')$ must have points in $F_{n-1}[\phi(P), \epsilon]$ that are also points of $\phi(L_1)$. Thus $\phi(L_4) \neq 0$.

Since $\phi(L_3) \cdot \phi(L_4) \subset F_{n-1}[\phi(P), \epsilon]$, and since Q is a point of $F_{n-1}[\phi(P), \epsilon]$ that is not in $\phi(L_1)$, it is clear that $\phi(L_3) \cdot \phi(L_4)$ does not fill up the surface $F_{n-1}[\phi(P), \epsilon]$, and consequently that the $(n-2)$th Betti number (mod 2)† of $\phi(L_3) \cdot \phi(L_4)$ is zero. In symbols,

$$(8) \qquad p^{n-2}[\phi(L_3) \cdot \phi(L_4)] = 0.$$

By (2), (6) and (7),

$$(9) \qquad L = L_1 + L_2 = L_3 + (L_2 + L_4).$$

* Anna Mullikin, *Certain theorems relating to plane connected point sets*, Transactions of this Society, vol. 24 (1922), pp. 144–162.

† See P. Alexandroff, *Une définition des nombres de Betti pour un ensemble fermé quelconque*, Comptes Rendus, vol. 184 (1927), pp. 317–319.

Neither of the sets L_3, L_2+L_4, is identical with L; for L_2+L_4 does not contain P, and L_3 contains no point of L_2. Consequently, since L is an *irreducible* cut between c and c_1,

$$(10)\qquad \begin{aligned} c + c_1 &\sim 0 \qquad (\text{mod } 2, E_n - L_3), \\ c + c_1 &\sim 0 \quad [\text{mod } 2, E_n - (L_2 + L_4)]. \end{aligned}$$

Now

$$(11)\qquad L_3\cdot(L_2 + L_4) = L_2\cdot L_3 + L_3\cdot L_4 = L_3\cdot L_4.$$

Since $L_3\cdot L_4$ is homeomorphic with $\phi(L_3)\cdot\phi(L_4)$, and the Betti number of a closed set is an analysis situs invariant, it follows from (8) and (11) that

$$(12)\qquad p^{n-2}[L_3\cdot(L_2 + L_4)] = p^{n-2}(L_3\cdot L_4) = 0.$$

Consequently, by virtue of Alexandroff's generalization of the Phragmén-Brouwer theorem,* and relations (9), (10), and (12),

$$(13)\qquad c + c_1 \sim 0 \qquad (\text{mod } 2, E_n - L).$$

But this is a contradiction of the fact that L is a cut of E_n between c and c_1. Thus the supposition that H contains a point not in $\phi(L_1)$ leads to a contradiction, and

$$(14)\qquad H \subset \phi(L_1).$$

From relations (3) and (14) we have that

$$(15)\qquad \phi(C) \subset \phi(L_1),$$

and hence, from (3) and (15),

$$(16)\qquad \phi(C) \subset H\cdot\phi(L_1).$$

From relations (4), (14), and (16) it follows that

$$\phi(C) = H\cdot\phi(L_1) = H.$$

As H is an open connected subset of E_{n-1}, the theorem is proved, the relations $\overline{H}-H=\phi(\overline{C}-C)=\phi(B)$ being an immediate consequence of the fact that $H=\phi(C)$.

THEOREM 2. *In E_n, let D be a bounded domain such that all*

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* P. Alexandroff, *Une généralization nouvelle du théorème de Phragmén-Brouwer*, Comptes Rendus, vol. 184 (1927), pp. 575–577.

*1-cycles of D are homologous to zero in D.** *Then if two points c_1 and c_2 of D are separated in D by $A \cdot D$, there is a component, C, of $A \cdot D$ which separates c_1 and c_2 in D.*

PROOF. Let that component of $D - A \cdot D$ which contains c_1 be denoted by G. Then $\overline{G} - G$ is a cut of E_n between c_1 and c_2, and contains an irreducible cut, L, between c_1 and c_2. Let L_1 and L_2 be defined as in (2′) above. Let P be a point of $L_1 \cdot D$, and let C be the component of $A \cdot D$ determined by P. Let $B = A \cdot F$, where F is the boundary of D, and let H be the component of $E_{n-1} - \phi(B)$ determined by $\phi(P)$. As before, if we suppose H contains a point which is not in $\phi(L_1)$, we can separate H according to relations (5), and proceed to a contradiction; consequently $H \subset \phi(L_1)$. Since $\phi(L_1) \subset \phi(A)$, we have $H \subset \phi(A)$ and accordingly $\phi^{-1}(H) \subset A$. Since we know that H contains no point of $\phi(B)$, $\phi^{-1}(H) \subset A - B$. Therefore $\phi^{-1}(H) \subset C$. It is clear that $\phi(C) \subset H$, and therefore $C \subset \phi^{-1}(H)$. Consequently $\phi^{-1}(H) = C$ and $H = \phi(C)$. Since, as noted above, $H \subset \phi(L_1)$, it follows that C is the component of $L_1 \cdot D$ determined by P. Thus every component of $L_1 \cdot D$ is homeomorphic with an $(n-1)$-dimensional domain of E_{n-1} whose boundary is in $\phi(B)$.

Let t be any 1-chain in D bounded by $c_1 + c_2$. Not more than a finite number of the components of $L_1 \cdot D$ contain points of t. For suppose infinitely many contain points of t, and let $C_1, C_2, C_3, \cdots$ denote these components; they form a denumerable collection, since, as just shown, every component of $L_1 \cdot D$ is homeomorphic with a domain of E_{n-1}, and no two components have points in common. Let x_i be a point of $C_i \cdot t$, $(i = 1, 2, 3, \cdots)$. Then the set $\sum_{i=1}^{\infty} x_i$ has at least one limit point, y, on t. As L_1 is closed, $y \subset L_1$, and there is a component, U, of $L_1 \cdot D$ that contains y. But $\phi(y)$ is an interior point of the domain $\phi(U)$ and cannot be a limit point of the set $\sum_{i=1}^{\infty} \phi(x_i)$. The contradiction

* We refer here to modulo 2 homologies. See J. W. Alexander, *Combinatorial analysis situs*, Transactions of this Society, vol. 28 (1926), pp. 301–329. The necessity for this condition on the 1-cycles is made evident by the case where D is the interior of the anchor-ring in E_3; for a plane may be passed through D, in this case, in such a way that two points of D are separated by two components of the plane section, but not separated by either one of the components. An important case where the condition is satisfied is of course that in which D is bounded by the topological $(n-1)$-sphere.

is obvious. Let those components of $L_1 \cdot D$ that have points in common with t be denoted by $K_1, K_2, \cdots, K_m$.

One of the components K_i separates c_1 from c_2 in D. To show this, we note first that their sum $\sum_{i=1}^m K_i$ separates c_1 from c_2 in D. For suppose this is not the case. Then, denoting the set $L_1 \cdot D - \sum_{i=1}^m K_i$ by L_1', there exist 1-chains T_1^1 $(=t$, say) and T_2^1, such that*

$$(17) \quad \begin{cases} T_1^1 \rightarrow c_1 + c_2, [D - L_1']; \ [E_n - (L_2 + L_1')], \\ T_2^1 \rightarrow c_1 + c_2, \left[D - \sum_{i=1}^m K_i\right]; \ \left[E_n - \left(L_2 + \sum_{i=1}^m K_i\right)\right]. \end{cases}$$

The sets $L_2 + L_1'$ and $L_2 + \sum_{i=1}^m K_i$ are closed, and have in common only L_2. The latter set, however, lies entirely in F. Accordingly, by the hypothesis, there exists a 2-chain, T^2, bounded by $T_1^1 + T_2^1$ in D, and we have

$$(18) \quad T_1^1 + T_2^1 \sim 0, \left[E_n - (L_2 + L_1') \cdot \left(L_2 + \sum_{i=1}^m K_i\right)\right].$$

By Alexander's Addition Theorem,† and by (17) and (18),

$$(19) \qquad c_1 + c_2 \sim 0, \quad (E_n - L).$$

But (19) contradicts the fact that L is a cut of E_n between c_1 and c_2. Consequently one of the sets L_1', $\sum_{i=1}^m K_i$, separates c_1 and c_2 in D, and as the former set has no points in t, it is obvious that c_1 and c_2 are separated by $\sum_{i=1}^m K_i$ in D.

The proof can now be completed by a finite number of steps. If K_1 does not separate c_1 and c_2 in D, we can show by use of the Alexander Addition Theorem that $\sum_{i=1}^m K_i$ separates c_1 and c_2 in D. By process of elimination we must finally arrive at a set K_j, $(1 \leqq j \leqq m)$, which separates c_1 and c_2 in D. As $K_j \subset L_1$ and $L_1 \subset A$, the component $K_j \subset A$ and the theorem is proved.

THEOREM 3. *Let P be a point of A. Then for any positive number ρ there exists a positive number ϵ such that if Q is any point of*

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* If C^i denote an i-cycle, then the relation $M^{i+1} \rightarrow C^i$ is to be interpreted "M^{i+1} is an $(i+1)$-chain bounded by C^i." See J. W. Alexander, loc. cit. All congruences and homologies used in the present instance are to be understood as modulo 2, without explicit statement of that fact in the relations given.

† J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, Transactions of this Society, vol. 23 (1922), pp. 333–349, Corollary W^i.

$(E_n-A)\cdot S_n(P, \epsilon)$, *then that component of* $(E_n-A)\cdot S_n(P, \rho)$ *which contains* Q *has* P *as a boundary point.*

PROOF. Suppose there exists a positive number ρ for which the theorem is not true. Let ϵ_1 be a positive number less than 1 as well as less than ρ. Then the neighborhood $S_n(P, \epsilon_1)$ contains a point, x_1, of E_n-A, such that the component, G_1, of $(E_n-A)\cdot S_n(P, \rho)$ which contains x_1 does not have P as a boundary point. Then $\overline{G}_1-G_1$ is a cut of E_n between x_1 and P. Accordingly, $(\overline{G}_1-G_1)\cdot A\cdot S_n(P, \rho)$ is a cut of $S_n(P, \rho)$ between x_1 and P, and by virtue of Theorem 2 (as applied to the closed set $(\overline{G}_1-G_1)\cdot A$) there is a component, C_1, of this set, which separates x_1 and P in $S_n(P, \rho)$. By Theorem 1 the set $\phi(C_1)$ is a domain of E_{n-1} whose boundary is $\phi(\overline{C}_1-C_1)\subset\phi[A\cdot F_n(P, \rho)]$. It is clear, then, that C_1 is a component of $A\cdot S_n(P, \rho)$.

Let ϵ_2 be a positive number less than $\frac{1}{2}$ as well as less than ϵ_1 and $\rho(P, C_1)$. Then $S_n(P, \epsilon_2)$ contains a point x_2 of E_n-A such that the component, G_2, of $(E_n-A)\cdot S_n(P, \rho)$ which contains x_2 does not have P as a boundary point. As before, there is a component, C_2, of $A\cdot S_n(P, \rho)$ which separates x_2 from P in $S_n(P, \rho)$.

Continuing in this way, we obtain a sequence of distinct points $x_1, x_2, x_3, \cdots$, having P as a sequential limit point, and a sequence $C_1, C_2, C_3, \cdots$ of distinct components of $A\cdot S_n(P, \rho)$ such that for every i, x_i and P are separated in $S_n(P, \rho)$ by C_i.

From the fact that $x_i\subset S_n(P, \epsilon_i)$, and C_i separates x_i and P in $S_n(P, \rho)$, it follows that there is a point y_i of C_i in $S_n(P, \epsilon_i)$. That the sequence $y_1, y_2, y_3, \cdots$ has P as a sequential limit point is obvious. Let

$$\overline{C}_i\cdot F_n(P, \rho) = B_i, \qquad (i = 1, 2, 3, \cdots). \tag{20}$$

Then $B_i\subset A$, and by Theorem 1, $\phi(C_i)$ is a domain of E_{n-1} whose boundary is $\phi(B_i)$.

For every i, $\phi(B_i)$ separates $\phi(y_i)$ from $\phi(P)$ in E_{n-1}, and hence $\phi(P)$ is a limit point in E_{n-1} of the set $\sum_{i=1}^{\infty}\phi(B_i)$. But then in E_n, P must be a limit point of the set $\sum_{i=1}^{\infty}B_i$, which is absurd since by (20) the sets B_i are all in $F_n(P, \rho)$.

The following theorem now follows simply from Theorem 3.

THEOREM 4. *Every point of* A *is regularly accessible from* E_n-A.

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