ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

58. Dr. L. S. Kennison: *Conformal transformations in function space.*

Conformal transformations of the space L_2 are defined and shown to form a group. The following analog of Liouville's theorem is proved: All conformal transformations in function space are composed of translations, rotations, transformations of similitude, and inversions. (Received January 19, 1932.)

59. Dr. L. S. Kennison: *Note on homogeneous functionals.*

The analog for functionals of Euler's theorem for homogeneous functions of n variables was proved by E. Freda in 1915. This note contains a proof of this theorem for a more general case, also a proof of the theorem, not proved by Freda, that if $F[\lambda\phi(x)]=K(\lambda)F[\phi(x)]$, then $K(\lambda)$ is a power of λ. (Received January 19, 1932.)

60. Professor J. H. Roberts: *Concerning spaces which are uniordered relative to systems of closed and compact point sets.*

G. T. Whyburn has raised the following question (see Fundamenta Mathematicae, vol. 16, p. 170): *Can every separable metric space S which is uniordered relative to a system Z be transformed by a biunivalued and continuous transformation into a separable metric space S^* in which every point P^* is contained in arbitrarily small neighborhoods with boundaries belonging to Z?* The space S is uniordered relative to Z if for every point P there exists a monotonic sequence of domains $U_1, U_2, \cdots$, containing P, each bounded by an element of Z, and such that $P=\Pi(\overline{U_i})$. By a "system Z" is understood a collection Z of closed point sets such that every closed subset of an element of Z is an element of Z, and the sum of every two elements of Z is an element of Z. An example shows that the answer to Whyburn's question is in the negative. However, it is proved that with the additional hypothesis that the elements of Z be *compact* the answer is in the affirmative. (Received January 15, 1932.)

61. Professor J. L. Walsh: *An expansion of meromorphic functions.*

If $f(z)$ is a meromorphic function (that is, analytic, except for poles, at every finite point of the plane) whose poles are included in the sequence $\alpha_1, \alpha_2, \cdots$, and if the points $\beta_1, \beta_2, \cdots$ are distinct from the α_n and uniformly limited, then the formal expansion of $f(z)$, $f(z) = a_0 + a_1(z-\beta_1)/(z-\alpha_1) + a_2(z-\beta_1)(z-\beta_2)/((z-\alpha_1)(z-\alpha_2)) + \cdots$, found by interpolation in the points β_n converges to $f(z)$ for all finite values of z except the α_n. In particular, if $\beta_1 = 0$, $\beta_{n+1} = 1/\bar{\alpha}_n$, and if $f(z)$ is analytic for $|z| \leqq 1$, then the sum of the first $n+1$ terms of this series is the function $(a_{0n}z^n + a_{1n}z^{n-1} + \cdots + a_{nn})/((z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n))$ of best approximation to $f(z)$ on the unit circle in the sense of least squares. (Received January 15, 1932.)

62. Professor J. L. Walsh: *Interpolation and functions analytic interior to the unit circle.*

If the product $\Pi|\beta_n|$ converges, $|\beta_n| < 1$, and if we have $f(z) = (1/(2\pi i)) \cdot \int_C (f_1(t)/(t-z))\, dt$, C: $|z| = 1$, where $f_1(z)$ is integrable together with its square on C, then the formal expansion of $f(z)$ in a series $f(z) \sim a_0 + a_1 z/(1-\beta_2 z) + a_2 z(z-\beta_2)/((1-\beta_2 z)(1-\bar{\beta}_3 z)) + \cdots$, found by interpolation in the points $\beta_1 = 0, \beta_2, \beta_3, \cdots$, or (what is equivalent) by expanding $f_1(z)$ formally on C in this series of orthogonal functions, converges for $|z| < 1$, uniformly for $|z| \leqq r < 1$. The function $\phi(z)$ represented coincides with $f(z)$ in the points β_n and can be expressed by $\phi(z) = (1/(2\pi i)) \int_C (\phi_1(t)/(t-z))dt$, where $\phi_1(t)$ is integrable together with its square on C. (Received January 16, 1932.)

63. Professor J. L. Walsh: *Note on the degree of convergence of sequences of analytic functions.*

If the sequence of functions $f_n(z)$, analytic in a region C, converges uniformly in C to the function $f(z)$ with a certain degree of convergence, and if the function $f_n(z) - f(z)$ is known to have a certain number of zeros at specified points interior to C, then Jensen's theorem yields a result on the degree of convergence of $f_n(z)$ in an arbitrary region interior to C. This remark is particularly applied to the study of series of the form $a_0 + a_1(z-\beta_1)/(z-\alpha_1) + a_2(z-\beta_1)(z-\beta_2)/((z-\alpha_1)(z-\alpha_2)) + \cdots$. (Received January 15, 1932.)

64. Professor J. L. Walsh: *On the overconvergence of sequences of rational functions.*

This paper considers more general situations on the overconvergence of sequences of rational functions than have hitherto been studied, and indeed seems to prove the most general theorem of its kind. (Received January 15, 1932.)

65. Professor J. L. Walsh: *On polynomial interpolation to analytic functions with singularities.*

Let $f(z)$ be continuous on C: $|z| = 1$, and let $p_n(z)$ denote the (unique) polynomial of degree n which coincides with $f(z)$ in the $(n+1)$st roots of unity. Then we have $\lim_{n\to\infty} p_n(z) = (1/(2\pi i)) \int_C (f(t)/(t-z))dt$ uniformly for $|z| \leqq r < 1$. (Received January 15, 1932.)

66. Dr. G. T. Whyburn: *A certain transformation on metric spaces.*

In this paper a study is made of the transformation T effected on a connected and locally connected metric space M by changing all distances $\rho(x, y)$ into distances $\rho^*(x, y) = \min [\delta(C)]$, where C is any connected subset of M containing $x+y$. It is shown, among other results, that T is a homeomorphism whose inverse is uniformly continuous and which leaves invariant diameters of connected sets and hence also Property S. Furthermore, $T(M)$ is uniformly locally connected. The results found are applied to show that if R is any simply connected plane region having Property S with boundary B, then the boundary J of $T(R)$ is a simple closed curve; and since the inverse of T is uniformly continuous, it can be extended in one and only one way to give a continuous transformation W of J into B. Furthermore, W is a minimal mapping, from the standpoint of multiplicity, of J onto B. The multiplicity under W of any point p of B is equal to the number of components of $B-p$ when this number is finite, and is infinite when this number is infinite. (Received January 8, 1932.)

67. Professor E. B. Stouffer: *A geometrical determination of the canonical quadric of Wilczynski.*

Canonical developments for the equation of a curved surface take their simplest forms when the vertices of the associated tetrahedrons of reference lie on the canonical quadric of Wilczynski. In the present paper this quadric is located geometrically in a natural and simple manner by means of its intersection with the axis of Čech. (Received January 20, 1932.)

68. Professors A. A. Albert and Helmut Hasse: *A determination of all normal division algebras over an algebraic number field.*

In a p-adic arithmetic treatment, H. Hasse has reduced the problem of proving that every normal division algebra over an algebraic number field F of finite degree is a cyclic (Dickson) algebra to the question of the validity of the following theorem: If a normal division algebra A of degree M (order m^2) over F splits everywhere, then $m=1$. Hasse also proved that this latter theorem is true if we assume that A is cyclic of degree unity, or a prime p. It is shown now how the above theorem is an immediate consequence of Hasse's arithmetic results and already published theorems of A. A. Albert. Also a new proof is given following the ideas of Albert's proof but with simplifications due to the omission of extraneous material. (Received January 20, 1932.)

69. Professor A. A. Albert: *On normal simple algebras.*

The author's theorems on the direct product of a normal division algebra and an algebraic field (Transactions of this Society, vol. 33 (1931), pp. 690–711) have proved to be very powerful tools for research in the theory of linear algebras. In the present paper the author throws new light on the meaning of the former theorems in a study of the properties of a subfield K of any normal simple algebra A relative to the division algebra component of A. Finally the results obtained are used to prove the validity of a conjecture of L. E.

Dickson (Transactions of this Society, vol. 28 (1926), pp. 227, 228) as to a necessary and sufficient condition that a normal simple algebra be a division algebra. (Received January 20, 1932.)

70. Professor E. P. Lane: *Surfaces and curvilinear congruences.*

The purpose of this paper is to begin the study of the projective differential geometry of the configuration composed of a surface and a curvilinear congruence in ordinary space, the points of the surface and the curves of the congruence being in one-to-one correspondence. This generalization from rectilinear to curvilinear congruences possesses interest because it not only contains many of the results of recent literature as special cases but also leads to new and more general results. Congruences of plane curves, one of which lies in each tangent plane of the surface, receive particular attention, and still more specially such congruences of conics and plane cubic curves are studied. (Received January 29, 1932.)

71. Professor E. B. Stouffer: *On the contact of two space curves.*

Associated with two space curves which have contact of order n at a point there are a principal plane discovered by Halphen and in general a principal line and a principal point discovered by Bompiani. In the present paper all three of these elements are obtained by a simple process involving only a linear transformation. The method may be extended to obtain similar results in hyper-space. (Received February 10, 1932.)

72. Dr. R. S. Burington: *A classification of quadrics in affine N-space by means of arithmetic invariants.*

The classification of real conics under the euclidean group has been discussed by MacDuffee, Paradiso, and Franklin, with the use of algebraic invariants (American Mathematical Monthly, vol. 33, pp. 243–252, pp. 406–418; vol. 34, pp. 453–467). In some of their euclidean canonical forms there are infinitely many quadrics, all of which are equivalent under the affine group, and not equivalent to a quadric of any other canonical form. Thus the problem of the separation into types is a problem in real affine geometry. It is the purpose of this paper to show that the matrix of the quadric has four arithmetic invariants under the real affine group which are sufficient to give a complete separation of the quadrics into types. The types obtained by this means coincide exactly with those previously obtained, and the labor in applying the theory to a given quadric is slight. The types of real quadrics for $n=2, 3,$ and 4 are listed in detail. (Received February 11, 1932.)

73. Professor A. A. Albert: *A construction of non-cyclic normal division algebras.*

A construction is given of two generalized quaternion algebras B and C over a function field $F(u, v)$ of all rational functions with coefficients in a real field F of two independent marks u and v. It is shown that the algebra $A = B \times C$ is a normal division algebra of order sixteen and is not a cyclic (Dickson) algebra. This is the first example of non-cyclic algebras in the literature, and the very important question as to their existence is finally settled. (Received February 20, 1932.)

74. Professor W. D. Baten: *Sampling from many parent populations.*

Out of the parent population A_i, consisting of n_i variates, r_i variates are taken at random. The sample considered in this article consists of r_1 variates from parent A_1, r_2 variates from parent $A_2, \cdots$, and r_s variates from parent A_s. All possible sample means are obtained. The distribution of these sample means is examined from the nature of the standardized moments when the number of variates in each parent is finite, when the number of variates approaches infinity, and finally when the numbers of variates taken from each parent approach infinity. General formulas are given for determining any moment by use of the Carver polynomials. This distribution of sample means is shown to approach the normal distribution when the parents contain an infinite number of variates and the number of variates taken at random from each parent approaches infinity, regardless of the nature of the parent populations. (Received February 24, 1932.)

75. Professor H. S. Wall: *On the Padé table for a power series having a corresponding continued fraction in which the coefficients have limiting values.*

Let $P(z)$ have a corresponding continued fraction $b_1/(1+(b_2z/1+(b_3z/1 + \cdots)))$ in which $\lim_n b_n = b$. Then if $b=0$ and $\lim_n(b_n/b_{n-1}) = \pi > 0$, every diagonal file of the Padé table converges to one and the same meromorphic limit. If $b \neq 0$ the files converge to a common limit over the entire plane except along the whole or a part of that segment of the real axis from $x = -\frac{1}{4}b$ to $x = \infty$ which does not contain the origin, and except possibly at certain isolated points. Within the plane so cut the limit is holomorphic except at these isolated points, which are poles. (Received March 1, 1932.)

76. Mr. Rufus Oldenburger: *Canonical binary trilinear forms.*

E. Schwartz obtained binary trilinear forms canonical under transformations in the complex field. These forms are here obtained directly from the theory of pairs of binary bilinear forms, and are characterized by two arithmetic invariants one of which is the generalization of ordinary matrix rank. (Received March 5, 1932.)

77. Professor C. C. Camp: *A new method for finding the numerical sum of an infinite series.*

$\sum_{n=1}^{\infty} u_n = s$, where u_n is a positive quantity, can be evaluated by defining u_n for all real $n \geqq 1$ as in the Cauchy integral test, adding the first n terms, and estimating the remainder R_n by formula (1): $\int_{n+1}^{\infty} u_n dn + \frac{1}{2}u_{n+1} < R_n < \int_n^{\infty} u_n dn - k_n u_n$, where $k_n = (\int_n^{n+2} u_n dn - u_{n+1} - u_{n+2})/(u_n - u_{n+2})$. We take u_n so that its derivative will approach zero monotonically. The upper bound is closer to R_n than the lower, the ratio of the deviations being about $1:p$ for $u_n = n^{-p}$. Accordingly, for $n=20$ one estimates s correct to 8 decimals for $p=2$. By using (1) for $n=5$ and $n=9$ for the series (2): $u_n = (n^{-5/2} + n^{-7/2}/4 + 21\ n^{-9/2}/128)/16(\pi n)^{1/2}$, which arises in the integration of $\int_0^1 (1-u^8)^{1/2}\ du$, one obtains upper bounds a_1, a_2 for s; and lower bounds b_1, b_2. Straight line extrapolation $t = b_2 + (b_2 - b_1)\Delta_2/(\Delta_1 - \Delta_2)$, where $\Delta_i = a_i - b_i$, $i=1, 2$, yields the sum correct to 7 decimals. Alternating series are converted into positive series by pairing terms. Stirling's series were used to obtain the asymptotic formula (2) and Simpson's rule to compute k_n. In summing Maclaurin's series for $\sin^{-1} 1$ similarly, $n=8$, 12 gave t with an error $<u_{13}/6000$. (Received March 5, 1932.)

78. Dr. L. M. Blumenthal: *Concerning regular pseudo d-cyclic sets.*

In a paper now in press (*A complete characterization of proper pseudo d-cyclic sets of points*, American Journal of Mathematics) the author has characterized those pseudo d-cyclic sets that (1) contain no convex tripod, and (2) contain no pseudo d-cyclic quadruples that are pseudo-linear. Such sets were called proper. The purpose of this paper is to characterize pseudo d-cyclic sets that contain no convex tripods, the second condition above not being retained. Such sets are called *regular*. The principal theorem of the paper proves that such sets are equilateral provided that they contain more than four points. (Received March 5, 1932.)

79. Mr. J. H. Kusner: *On continuous curves with cyclic connection of higher order.*

J. R. Kline has raised the question whether, for $n>2$, there exists a continuous curve C such that every two points of it are joined in C by exactly n arcs which are mutually exclusive except for end points. It is here shown that for $n = 3$ no such curve exists. If, for any $n>2$, such a curve existed, it would have the property that every three of its points lie on a simple closed curve in it. The question arises whether there exist regular curves which have this property. The Sierpinski triangle curve is obviously of this type, and it is shown that any finite number of its points lie on a simple closed curve in it. A continuous curve C such that any n points of it lie on a simple closed curve in C is said to be n-cyclicly connected, in extension of the notion of cyclic connection introduced by Whyburn in 1927. Necessary and sufficient conditions for a continuous curve to be 3-cyclicly connected are derived, and some of the properties of such curves are developed. (Received March 5, 1932.)

80. Dr. Leo Zippin (National Research Fellow): *Irreducible continuous curves.*

It is proved that in order that a self-compact one-dimensional subset K of a generalized continuous curve (quasi-Peanian space) belong to a subcontinuous curve C'' of C which is *irreducible about* K, it is necessary and sufficient that (1) the components of K be continuous curves of points; (2) for every n, the components of K of diameter $1/n$ be in finite number. As special cases: whenever the curves of K belong to the class of acyclic, regular, perfect, or rational curves respectively, then C'' belongs to the same class. Ours is essentially an extension of an old plane theorem of Gehman. The restriction upon the dimension of K is occasioned by our proof, and is probably not required by the theorem. (Received March 3, 1932.)

81. Dr. A. B. Brown: *On Morse's duality relations for manifolds.*

Duality relations have been proved by Marston Morse (National Academy Proceedings, vol. 13 (1927), pp. 813–817) for cases which may be described, except for certain analytic restrictions on the configurations involved, as that involving the Betti numbers of a manifold with regular boundary and those of its boundary, and that involving the Betti numbers of a sub-complex of an absolute n-manifold, and the Betti numbers of the residual set. In the present paper, with the aid of duality relations of Alexander and Lefschetz, we establish the duality relations without analytic restrictions, and in the second case remove certain other restrictions. Certain additional duality relations are obtained. Shortly before writing this abstract, the writer became acquainted with the contents of a paper by E. R. van Kampen (Amsterdam Proceedings, vol. 31 (1928), pp. 899–905), which contains results of which Morse's second set of relations are a corollary, proved (in outline) in addition under broader hypotheses, and with additional results regarding intersection numbers. However, the proofs of the present paper are of simpler character, and in addition the author obtains certain duality relations not obtained by van Kampen. (Received March 4, 1932.)

82. Dr. Hassler Whitney (National Research Fellow): *Regular families of curves.* I.

A function of closed sets $\mu(S)$ is defined, such that if every point of S' is within ϵ of S, then $\mu(S') \leqq \mu(S)+2\epsilon$, and if $S \subset S'$, $S \neq S'$, then $\mu(S') > \mu(S)$. A family of curves is regular if, roughly, neighboring arcs of curves have a small écart distance. By using the function μ, a function $p'=g(p, \alpha)$ is defined, p' lying on the curve through p, which is continuous in both variables. The function also enables one to draw a cross section through any point of the family. Examples show the usefulness of the function in topology. (Received March 3, 1932.)

83. Dr. Hassler Whitney (National Research Fellow): *Regular families of curves.* II.

Let a regular family of curves form an open subset R' of a compact metric space R; the rest of R shall consist of "invariant points." It is shown how a sequence of "tubes" with certain properties fills out R'. A function $p'=f(p, t)$ is found with the following properties: (1) $f(p, t)$, $-\infty < t < +\infty$, forms exactly the curve through p, or is p, if p is an invariant point; (2) $f(p, t)$ is continuous in both variables everywhere; (3) $f[f(p, t_1), t_2]=f(p, t_1+t_2)$; also $f(p, 0)=p$. (Received March 3, 1932.)

84. Professor W. A. Wilson: *A relation between metric and euclidean spaces.*

This paper deals principally with convex complete metric spaces which have the property that any four points are congruent with four points in euclidean space. Along with other results it is shown that if such a space is also separable and externally convex, it is congruent with some euclidean space or with Hilbert space. (Received March 2, 1932.)

85. Professor W. A. Wilson: *On angles in metric spaces.*

In this note conditions are given which are sufficient for the definition of a system of angles in certain metric spaces analogous to angles in euclidean spaces. Tangents to simple arcs are defined and an intrinsic condition is found, which is necessary and sufficient for the existence of a tangent in the spaces discussed. (Received March 1, 1932.)

86. Mr. Abraham Sinkov: *Families of groups generated by two operators of the same order.*

Except in a few special cases, a group generated by two operators is not completely determined by their orders and the order of some additional combination of them. The set of groups which is obtained as a solution to such conditions quite often forms what may be considered a family in the sense that the groups are all of the same type and have their orders expressible as an algebraic function of one or more parameters. The present paper obtains the defining relations and exhibits the generating operations of two simply isomorphic families of solvable groups whose orders involve three parameters. The order is given by the expression nm^{n-k} where k is an arbitrary factor of n; it includes as special cases a few of the families which have previously appeared in the literature. (Received February 29, 1932.)

87. Mr. Marshall Hall: *Significance of quadratic residues in factorization.*

This paper deals with the so called eventual quadratic residues of unknown composition (Kraitchik's notation). The laws governing their combinations are considered, and the correspondence between a number and its set of eventual residues is shown. A theoretical test for primality is given, and also a modified form for practical applications. An example is given in illustration. (Received March 5, 1932.)

88. Professor Oystein Ore: *Non-commutative polynomials.*

This paper contains a theory of polynomials in a non-commutative field K. The coefficients are usually not commutative with the variable x, but the condition is imposed that the degree of a product is equal to the sum of the degrees of the factors. By means of the notion of transformation, the polynomials are classified in types, and three normal representations of polynomials are obtained. One of them is the decomposition in prime factors, which is usually not unique, and it may contain an infinite number of different factors. Among the applications of this theory are the formal theory of linear difference and differential equations, and the theory of equations in linear algebras. (Received March 3, 1932.)

89. Dr. J. L. Dorroh (National Research Fellow): *On the factorization of divisors of zero in finite commutative rings.*

This paper treats only finite rings which contain an identity under multiplication. Two divisors of zero are called associates of each other if each divides the other. A *prime* divisor of zero is one whose only factors are its associates and regular elements. It is known that a ring of the type considered is the direct sum of irreducible subrings and that the question of uniqueness in the decomposition of its divisors of zero into prime factors reduces to the same question for its irreducible subrings. The order of an irreducible finite ring is a power of a prime integer. It is shown that if the order of an irreducible finite ring R is p^k, then R contains just $p^{(r-1)k/r}$ divisors of zero, where r is a factor of k. Decomposition in R is unique if, and only if, R contains a nilpotent element of index r. Decomposition is unique in any finite ring whose order is not divisible by the cube of a prime integer. Some other types of finite rings are classified according to the uniqueness or ambiguity of the decomposition of their divisors of zero into prime factors. (Received March 4, 1932.)

90. Dr. Leonard Carlitz: *On factorable polynomials in several indeterminates over a Galois field.*

By a factorable polynomial is meant one that splits into linear factors in *some* Galois field. In this paper, a study is made of the distribution of factorable polynomials, particularly of irreducible polynomials; the expression obtained for the number of primary irreducible polynomials is a simple generalization of Gauss's formula for the case of a single indeterminate. Definitions are then framed for certain "arithmetic" functions of the polynomials; exact expressions are obtained for the sum of each of these functions taken over all polynomials of fixed degree. (Received March 2, 1932.)

91. Dr. Leonard Carlitz: *On an arithmetical functional equation.*

This paper is concerned with the functional equation $\sum\psi(d, x)\psi(\delta, y) = \psi(\nu, xy)$, the summation being taken over all d, δ with L.C.M. equal to ν. The equation is suggested in another paper of the author's (*On factorable polynomials in several indeterminates over a Galois field*) for the case $\psi(\nu, x) = \sum\mu(\nu/\delta)x^\delta$, summed over all δ dividing ν. The general equation is easily shown to be equivalent to a simple equation involving an arithmetic function of a single variable, whence the particular case above is very easily proved. It should be remarked that the second argument in $\psi(\nu, x)$ need not be integral. (Received March 2, 1932.)

92. Dr. Leonard Carlitz: *On a problem in additive arithmetic. Second paper.*

In a paper in the Quarterly Journal of Mathematics (Oxford Series, vol. 2 (1931), p. 97) the author, using the Hardy-Littlewood method of Farey dissection, obtained an asymptotic expression for $\sum\alpha(n_1)\cdots\alpha(n_\nu)$, summed over all partitions of n into $\nu(\nu\geqq 3)$ positive integers, $\alpha(n)$ being a factorable function satisfying certain conditions. In the present paper the more general sum, $\sum\alpha_1(n_1)\cdots\alpha_\nu(n_\nu)$, is considered, the $\alpha_i(n)$ being similar to $\alpha(n)$; by using an elementary method an asymptotic expression is found for *all* values of ν. (Received March 2, 1932.)

93. Professor Raymond Garver: *On the approximate solution of certain equations.*

A modification of Newton's method is presented which gives exact results for certain exponential equations instead of linear equations, and which gives very close approximations for many equations representing graphically similar functions. It is applied to the equation $x^x=c$, to the extraction of roots, and to other equations. (Received March 4, 1932).

94. Dr. B. F. Kimball: *The application of Bernoulli functions of negative order to differencing.*

In this paper the author calls attention to the formula (1) $\Delta^n f(x)=\sum_{m=0}^{\infty}(1/((2m)!))B_{2m}^{-n}(-n/2)f^{(n+2m)}(x+n/2)$ as useful in studying the differences of a function (compare Nörlund, *Differenzenrechnung*, 1924, p. 142). It is shown that, for $n>m$, $B_{2m}^{-n}(-n/2)$ takes the form of a polynomial of degree m in n with coefficient of leading term in n equal to $(2m)!/((24)^m(m!))$. Several applications of (1) are given, among which is the determination of the asymptotic value of $\Delta^n \log x$ as n becomes infinite, x positive. This is shown to be $(\Gamma(x))/(n^x \log n)$. The results can be modified to apply to the case where difference intervals are equal to a real quantity w, rather than equal to 1. (Received March 5, 1932.)

95. Professor C. N. Moore: *On certain sufficient conditions that a set of constants should be Fourier constants.*

The relationship between functions whose squares are integrable (L) and their corresponding Fourier constants is extremely simple and elegant. A combination of the theorems of Parseval and Riesz-Fischer tells us that the convergence of the series $\sum(a_n^2+d_n^2)$ is both a necessary and a sufficient condition that the function from which these constants have been obtained should be of integrable square. For the more general case of functions integrable (L) we have no such simple result. A necessary condition is that $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0$, but this is not sufficient. In the present paper it is shown that for the cosine series the conditions (1) $a_n \log n$ remains bounded, (2) $\sum a_n|\Delta^2 a_n|$ converges, are sufficient. These conditions are related to certain sufficient conditions derived by W. H. Young and Szidon, but they include cases of interest not covered by the theorems of the writers mentioned. (Received February 19, 1932.)

96. Professor Edward Kasner: *Successive curvatures of dynamical trajectories.*

A familiar theorem states that the radius of curvature r of a trajectory varies inversely as the square of the speed v. The present author gives the extension of this result, for any field of force, to the successive radii $r_1, r_2, \cdots$ of the successive evolutes of the trajectory. A separate discussion has to be made for the case of initial motion in the direction of the force, since then $r=\infty$. The results are then stated in terms of curvature γ and its derivatives γ_s, γ_{ss}, γ_{sss}, $\cdots$. In the simplest case, γ varies as v^{-2}. A separate theory is also needed when a particle starts from rest, as considered in earlier papers by the author (see Princeton Colloquium Lectures, p. 9, for the simplest theorem, curvature of trajectory equals one third the curvature of line of force, and the application by Roever, this Bulletin, vol. 21 (1914–15), p. 444). (Received March 5, 1932.)

97. Professor J. M. Thomas: *Regular differential systems of the first order.*

The theorem that every passive regular differential system of the first order is completely integrable follows readily from a result of Riquier's (*Les Systèmes d'Equations aux Dérivées Partielles*, p. 350). The present paper, however, shows that it can be obtained from Cauchy's existence theorem by direct generalization of a method due to J. Koenig (Mathematische Annalen, vol. 23), without any new convergence demonstration being necessary. The theorem is moreover connected with Cartan's existence theorem for non-singular solutions of Pfaffian systems. The relation between Cartan's and Riquier's work is thus established. Cartan's result is also generalized to systems whose left members are symbolic forms of higher degree than the first. (Received March 5, 1932.)

98. Professor A. K. Mitchell: *On a matrix differential operator.*

H. W. Turnbull (*On differentiating a matrix*, Proceedings of the Edinburgh Mathematical Society, (2), vol. 1, Part 2 (1928)) has defined a matrix differential operator and given several theorems concerning its properties. In this note simplified direct proofs of four of these theorems are given, the simplification resulting from the use of tensor notation and the properties of the generalized Kronecker delta. Finally, the effect of the operator upon the coefficients of the characteristic equation of a matrix is considered. (Received February 12, 1932.)

99. Professor B. F. Kimball: *Three theorems applicable to vibration theory.*

In discussions of the theory of free vibrations it is sometimes stated without proof that "the fact that the period of vibration is independent of the amplitude shows that the restoring force is linear." The present paper includes a proof of this statement and a brief consideration of the case of a non-linear restoring force of a fairly general nature. The origin is taken as the position of equilibrium; the force considered is a continuous function, positive with positive derivative for x positive. The force is zero at the origin with positive unilateral derivative there. It is taken symmetrical with respect to the origin. Under these conditions on the force, the statement quoted above is proved. We also establish the double inequality $4\cdot 2^{1/2}[a/f(a)]^{1/2}<T<4\cdot 2^{1/2}[a^2/F(a)]^{1/2}$, where T denotes the period of vibration, a the amplitude, $f(x)$ the force, and $F(a)=\int_0^a f(x)dx$. Adding to the above hypothesis the existence of unilateral second derivatives at the origin, we prove that the limiting value of the period as the amplitude of vibration approaches zero is $2\pi[f'(0)]^{-1/2}$. (Received March 1, 1932.)

100. Dr. S. S. Wilks (National Research Fellow): *Certain generalizations in the analysis of variance.*

By means of the distribution of the ratio of two independently distributed variances, R. A. Fisher has found, among other things, the distributions of the correlation ratio and of the multiple correlation coefficient in samples from a normal population of one variable in which these quantities are zero. Certain limiting forms of this distribution yield the χ^2 and Student's distributions. In the present paper this theory is extended to samples from a multivariate normal population, and the moments and distributions are found of the generalized forms of the variance, correlation ratio, and ratio of variances. Other results reached by the methods used in this paper include the moments and distributions of determinants and ratios of determinants of correlation coefficients, moments of the extension of the Pearson and Neyman λ criterion appropriate to testing the hypothesis that k samples are from the same multivariate normal population, and a new proof of Hotelling's generalization of Student's distribution. Moments of these statistics were found from Wishart's generalized product moment distribution (Biometrika, vol. 10A (1928)), and the expressions for their distributions were derived from the solutions of two integral equations. (Received February 24, 1932.)

101. Professor H. V. Craig: *Applications of a covariant differentiation process.*

This paper deals with certain invariants which are formally quite similar to the gradient, divergence, and curl. These invariants are defined by means of the process given in the paper entitled *On a covariant differentiation process*, this Bulletin, vol. 37 (1931), p. 731. (Received March 5, 1932.)

102. Dr. J. L. Doob: *The boundary values of analytic functions.*

If $f(z)$ is a function analytic and bounded for $|z|<1$, $\lim_{r\to\infty}f(re^{it})$ exists for almost all t on the interval $(0, 2\pi)$, defining the Fatou boundary function $F(z)=F(e^{it})$ almost everywhere on $|z|=1$. Let (1) $f_1(z), f_2(z), \cdots$ be a uniformly bounded sequence of functions analytic for $|z|<1$, with Fatou boundary functions (2) $F_1(z), F_2(z), \cdots$ respectively. Necessary and sufficient conditions on the sequence (2) are found that the sequence (1) converge uniformly in every closed subregion of $|z| < 1$. Let $f(z)$, $F(z)$ be as above, and suppose in addition that $f(z)$ is smooth. Then if P is a point on $|z| = 1$, necessary and sufficient conditions on $F(z)$ in a neighborhood of P are found (a) that $F(z)$ be defined at P, (b) that $f(z)$ have a given complex number as one of its limit values at P when z approaches P from within $|z| < 1$, (c) that this limit value be a limit value when z approaches P on a sequence of points lying in an angle whose sides are chords of $|z| < 1$ meeting at P. (Received February 6, 1932.)

103. Dr. J. J. Gergen: *Convergence criteria for double Fourier series.*

In this paper generalizations to two variables are given of certain familiar criteria for the convergence of a simple Fourier series: they are, chiefly, the analogues of the various forms of the Lebesgue test and the Hardy-Littlewood test. The tests for two variables are strictly analogous in so far as the characteristic conditions are concerned, but the continuity conditions are more restrictive than in the original tests. The relations between the generalizations to two variables of the six commonly recognized criteria for a single variable are discussed. It is shown, in particular, that the Lebesgue test includes Tonelli's test for double series, which deals with functions of bounded variation. (Received March 4, 1932.)

104. Professor George Rutledge: *The inverse matrix for de la Vallée-Poussin summation.*

The equations $s_n' = \mu_{n0}u_0 + \mu_{n1}u_1 + \cdots + \mu_{nn}u_n$, defining the de la Vallée-Poussin sequence for the series $u_0 + u_1 + u_2 + \cdots$, are solved for $u_0, u_1, u_2, \cdots$ by application of a relation between de la Vallée-Poussin summation and the Stirling interpolation series (Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 9 (1930), p. 261). (Received February 23, 1932.)

105. Professor George Rutledge: *A reliable method of obtaining the derivative function from smoothed data of observation.*

The essential tool in the process discussed in this paper is the interpolation polynomial (Lagrange) with coefficients in explicit form (Transactions of this Society, vol. 26 (1924), p. 113, and vol. 31 (1930), p. 807). Each value of the derivative function is obtained by retaining the figures common to three determinations the errors in which are normally not all of the same sign. Precise limits of error are established. The process has been used with marked success for obtaining the derivative function dp/dt for the function defined by the vapor pressure values of steam (F. G. Keyes and L. B. Smith, *Some final values for the properties of saturated and superheated water*, Mechanical Engineering, vol. 53 (1931), p. 132). (Received February 23, 1932.)

106. Dr. S. S. Wilks (National Research Fellow): *Estimates, moments, and distributions of certain statistics in fragmentary samples.*

In sampling from a multivariate population, it frequently happens that all of the items of a sample are not observed or classified with respect to all of the variates. In such cases questions arise concerning methods of estimating the population parameters from all of the observed data. In this paper, samples of N items are considered from a normal population of two variates x and y, in which s of the items are observed with respect to both x and y, m with respect to x only, and n with respect to y only. For given estimates of the means, approximations are obtained for the maximum likelihood estimates of the correlation coefficient and standard deviations of x and y in the population. The limiting forms of the variances and covariances of these estimates for large samples are found. Another system of estimates studied consists of those derived by maximum likelihood for each of the means, variances, and covariances, independently of the others. The characteristic function and an expression for the simultaneous sampling distribution were found for these estimates. (Received February 24, 1932.)

107. Dr. J. L. Doob: *Sequences of meromorphic functions.*

If $f(z)$ is a function meromorphic for $|z|<1$, we define its "cluster boundary function" $\mathfrak{F}(z)$ at each point of $|z|=1$ as the set of all limit values of $f(z)$ when z approaches that point from within $|z|<1$. Let (1) $f_1(z), f_2(z), \cdots$ be a sequence of functions meromorphic for $|z|<1$, with cluster boundary functions $\mathfrak{F}_1(z)$, $\mathfrak{F}_2(z), \cdots$ respectively. Let $\{A_n\}$ be a set of open or closed arcs on $|z|=1$. A point α will belong to the set s if there is a subsequence $\{f_{a_n}(z)\}$ and a sequence of points $\{z_{a_n}\}$ in $|z|<1$ such that $f_{a_n}(z_{a_n})\rightarrow\alpha$ and such that under the transformation $z'=(z-z_{a_n})/(\bar{z}_{a_n}z-1)$ the arc A_{a_n} is transformed into an arc A'_{a_n} on $|z|=1$ such that $\lim_{n\rightarrow\infty} mA'_{a_n}=2\pi$. A point α will belong to the set S if there is a subsequence $\{\mathfrak{F}_{a_n}(z)\}$ and a sequence of points $\{z_{a_n}\}, z_{a_n}$ on the arc A_{a_n}, such that $\mathfrak{F}_{a_n}(z_{a_n})\rightarrow\alpha$, for some determination of $\mathfrak{F}_{a_n}(z_{a_n})$ for each value of n. If α is a point of s, but not a point of S, and if D is the domain containing α whose frontier points all belong to S, it follows that $D\subset s$ and that every point of D, except possibly two, is assumed by an infinite number of the functions (1). If there are two exceptions in D, every point except these two in the extended plane is assumed by an infinite number of the functions (1). (Received February 6, 1932.)

108. Mr. W. V. D. Hodge: *The Dirichlet problem for harmonic functionals.*

In this paper it is shown that if, in a euclidean n-cell with a regular boundary, we consider functionals of cycles of q dimensions which satisfy Volterra's condition for harmonic functionals, we can extend the methods of potential theory to show that such functionals are uniquely determined by assigned values on the q-cycles of the boundary, when these values are suitably restricted in respect to continuity. It is also shown that it is possible to define harmonic functionals of the $(p-1)$-dimensional bounding cycles on a real closed analytic variety of $2p$ dimensions, which is subject to certain restrictions. It is proved that there are exactly R_p linearly independent functionals of the cycles which are harmonic everywhere on the variety, where R_p is the pth Betti number of the variety. (Received March 4, 1932.)

109. Mr. W. S. Lawton: *On the zeros of certain polynomials related to Laguerre and Jacobi polynomials.*

The object of this paper is to study the distribution of the zeros of the polynomials defined as follows: $J_n(x, \alpha, \beta)=x^{1-\alpha}(1-x)^{1-\beta}(d^n/dx^n)\ [x^{n+\alpha-1}\cdot(1-x)^{n+\beta-1}]$, $L_n(x, \alpha)=x^{1-\alpha}e^x(d^n/dx^n)\ [x^{n+\alpha-1}e^{-x}]$ for $\alpha, \beta \leqq 0$. (If $\alpha, \beta > 0$, these polynomials are known as Jacobi and Laguerre polynomials respectively, with all the zeros in the respective intervals (0, 1), (0, ∞).) As an illustration of the results obtained, we cite the following: if p and q are positive integers such that $0<\alpha+p$, $\beta+q<1$, then $J_n(x, \alpha, \beta)$ has *exactly* $n-p-q$ zeros inside the interval (0, 1), and a similar result holds for $L_n(x, \alpha)$. (Received February 25, 1932.)

110. Mr. S. B. Myers: *Sufficient conditions in the problem of the calculus of variations in n-space in parametric form and under general end conditions.*

Sufficient conditions for a minimum in this general problem in parametric form are given here. The results are in terms of the characteristic roots of a linear boundary problem, and are in close relation to the conditions recently given by Morse in the corresponding problem in non-parametric form. An important feature of the results is that the usual "non-tangency" hypothesis is not made. For example, if these results were applied to the problem of minimizing an integral along curves joining a point to a manifold, we would obtain sufficient conditions for a minimum even in the case that the minimizing curve is tangent to the manifold. The essential idea in the methods used in the paper is the treatment of the parametric problem as the limiting case of a series of regular non-parametric problems by means of a suitable modification of the integrand. (Received February 27, 1932.)

111. Mr. Jacob Sherman: *On the numerators of the convergents of the Stieltjes continued fraction.*

This paper deals with the numerators of the convergents of the continued fractions "associated" with and "corresponding" to the Stieltjes integral $\int_a^b(1/(x-y))d\psi(y)$, $\psi(y)$ bounded and non-decreasing in the interval (a, b). Its main result states that *the numerators of both the even and odd convergents of the corresponding continued fraction*, the former being equal to the numerators of the associated continued fraction, *constitute systems of orthogonal polynomials.* For the associated continued fraction, the denominators of whose convergents are the classical orthogonal polynomials (of Jacobi, Laguerre, and Hermite), there is derived a linear differential equation, of the second order, for the numerators closely related to that for the denominators, including the result of Christoffel for the Legendre numerators as a very special case. Methods are given for the expansion of the numerators in terms of the denominators with explicit formulas for the cases of Laguerre and Hermite. A relation is obtained between the characteristic function, $\psi(y)$, for the denominators and numerators, and also for other related functions dealt with in the theory of orthogonal Tchebycheff polynomials. The paper closes with some asymptotic evaluations for these new systems of polynomials. (Received February 18, 1932.)

112. Professor B. H. Brown: *Equiareal maps with parabolic meridians and parallels.*

In this paper the following problem is solved: to find all the area-preserving transformations in the plane which carry the rectilinear coordinate lines into a system of straight lines or parabolas with axes in a fixed direction. There are 21 distinct types of transformations, of which 8 are parabola-parabola systems, 11 line-parabola systems, and 2 line-line systems. Of these, 19 systems have the two families of different types, and 2 have them equivalent. There are thus 40 different possible maps. The general transformations are complicated, but can all be given explicitly in terms of algebraic functions. Only 3 of the 21 types appear to be old. Several of the new types seem well adapted for cartographic use. This is the first paper of a series dealing with general problems in area-preserving transformations. (Received February 20, 1932.)

113. Mr. George Comenetz: *The limit of the ratio of arc to chord for the curves drawn at a point on a surface.*

At an analytic point of a curve in complex space of three dimensions, the limit of the ratio of arc to chord, called the *rac*, is usually unity, but in special cases it can have any value whatever. Instead of the complete set of curves at a point in space, consider only the set of curves which lie on a surface analytic at the point. In the current issue of the Proceedings of the National Academy of Sciencies, Professor Kasner shows that the corresponding *rac set* will be one of five kinds, depending on the surface: (A) every complex value; (B) every value, with one real exception (the general case); (C) a discrete set of real values: 1.00, 0.94, 0.86, $\cdots$ (as in the ordinary plane); (D) unity and one other real value; (E) unity alone (as in a minimal plane). In a separate paper we shall give a geometrical classification of the curves at a point of a surface with respect to the value of the rac. The tangent plane at the point of the surface is of course either non-minimal or minimal; this distinction turns out to be a natural one for the problem. (Received February 10, 1932.)

114. Dr. S. S. Cairns: *The direction cosines of a p-plane, L_p, in a euclidean n-space, S_n.*

Let $(y_1, \cdots, y_n)$ denote rectangular cartesian coordinates in S_n. Consider a p-dimensional volume, V, on L_p. If the projection of V on the $(y_{i_1}, \cdots, y_{i_p})$-plane $(i_1<i_2< \cdots <i_p)$ is a p-dimensional volume, $V_{i_p\ldots i_1}$, let $\gamma_{i_1\ldots i_p} = V_{i_p\ldots i_1}/V$. Otherwise let $\gamma_{i_1\ldots i_p}=0$. The $\binom{n}{p}$ ratios $\gamma_{i_1\ldots i_p}$ will be called direction cosines of L_p. The sum of the squares of the γ's is unity. In all, there are $\binom{n}{p}-n(n-p)$ independent relations among the γ's, analogous to Vahlen's

identities (*Ueber die Relationen zwischen den Determinanten einer Matrix.* Journal für Mathematik, vol. 112 (1893)). For, if L_p is represented by $y_i = \Sigma_{j=1}^{p} a_{ij}u_j + y_i^0 (i=1, \cdots, n)$, then $\gamma_{i_1 \ldots i_p} = | \delta_{i_1 \ldots i_p}/\Delta |$, where $\delta_{i_1 \ldots i_p}$ is the determinant of the rows $i_1, \cdots, i_p$ of the matrix $\|a_{ij}\|$, and Δ^2 is the sum of the squares of the δ's. The following theorem on matrices underlies the above results: Let $A_{st} = \Sigma_{i=i}^{n} a_{is}a_{it} (s=1, \cdots, p;\ t=1, \cdots, p)$. Then $\Delta^2 = \Sigma' \pm (A_1 \cdots, A_{pi_p})$, where Σ^1 is a summation over all permutations $(j_1, \cdots, j_p)$ of $(1, \cdots, p)$, the sign being plus for even and minus for odd permutations. If, in particular, $A_{st}=1$ or 0, according as $s=t$ or $s \neq t$, then $\Delta^2 = 1$. (For $n=p$, the last sentence is a known theorem on determinants.) (Received March 5, 1932.)

115. Mr. J. M. Clarkson: *Some involutorial line transformations interpreted as points of V_2 of S_5.*

We consider four arbitrary pencils (A_i, α_i) of lines, in planes α_i, vertices A_i. A transversal line t determines a unique line of each, and these four lines have a second transversal t'. The involutorial transformation t, t' is interpreted in terms of points of a quadric hypersurface in S_i, and its properties are determined. Similarly, two pencils and a quadratic regulus define another involution; a third is obtained by taking two independent reguli. The images of ruled surfaces, congruences, and complexes are determined in each case. (Received February 25, 1932.)

116. Professor Louise D. Cummings: *Heptagonal systems of eight lines in a plane.*

The present investigation concerns the determination of the non-equivalent systems of eight real lines in a plane when no three of the lines are co-punctual. Two systems are equivalent if a one-to-one relation exists between the lines and the polygons of the two systems. This paper is limited to that subdivision of the problem where, in a set of eight lines, some seven form a convex heptagon. Each line is uniquely defined by its *mark*, which shows the arrangement of contiguous segments on the line. The eight marks are determined for each system and are tabulated together with the numbers of polygons of 3, 4, 5, 6, 7, 8 sides which occur in the system. The marks of the system are used to establish the non-equivalence of the eleven systems derived in this paper, and to determine the substitutions which transform the system into itself. (Received March 1, 1932.)

117. Professor Malcolm Foster: *On the inversion of a special cyclic system.*

This paper is concerned with the following cyclic system lying in the xz-planes of the moving trihedral of a surface S: $(x-R)^2+z^2=R^2$, $y=0$. If these circles are inverted relative to the vertex of the trihedral, we obtain a rectilinear congruence $x=k^2/(2R)$, $y=0$. The paper discusses the condition that the rectilinear congruence be also normal, and shows that two of the focal points of the cyclic system are collinear with one focal point of the rectilinear congruence. The inversion of the cyclic system $x^2+(z-R)^2=R^2$, $y=0$, is also considered. (Received March 3, 1932.)

118. Professor P. F. Smith: *On planar element strips.*

A strip of planar elements is a union of planar (surface) elements along a curve as defined by Sophus Lie. The differential geometry (classical) of element strips in space includes that of curves in spaces, and by it many theorems for surfaces are established. An element strip has three characteristic functions by which it is determined except as to position in space. Equations in series form for a strip are obtained, and also equations involving quadratures for certain special strips. Simultaneous differential invariants are derived for a strip and a surface upon which the strip lies. Two moving trihedrals (the tangent trihedral and the conjugate trihedral) are employed to give dual theorems. Spheres on three (four) consecutive points of strips are studied, and, dually, spheres on three (four) consecutive planes. The problem of two associated strips is discussed, and two examples (dual) when the strips have for corresponding elements a common center and radius of first (second) curvature are solved. (Received February 20, 1932.)

ERRATUM

Volume 37, page 818, abstract No. 343 (by L. E. Ward): substitute for the last sentence the following: "Conditions are given for the formal series for a function to converge uniformly to that function." The form in which this sentence appeared was due to clerical error.