NOTE ON DIOPHANTINE AUTOMORPHISMS*

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1. *Introduction.* Professor E. T. Bell‡ defines a diophantine automorphism of a form $f \equiv f(x_1, \cdots, x_n)$, where f is a homogeneous polynomial, to be a transformation of f into f^h, h an integer >1, by means of a substitution $X_i = X_i(x_1, \cdots, x_n)$, $(i = 1, \cdots, n)$, so that

$$(1) \qquad f(X_1, \cdots, X_n) = f^h(x_1, \cdots, x_n).$$

The first non-trivial instance of an algebraic phenomenon of this kind is due to Eisenstein,§ who noticed that the discriminant of the general binary cubic possesses such an automorphism. Indeed, if we set

$$f(x_1, x_2, x_3, x_4) = x_1^2 x_4^2 - 3x_2^2 x_3^2 + 4x_1 x_3^3 + 4x_2^3 x_4 - 6x_1 x_2 x_3 x_4,$$

$$X_1 = 3x_1 x_2 x_3 - x_1^2 x_4 - 2x_2^3 = -\frac{1}{2}\frac{\partial f}{\partial x_4},$$

$$X_2 = 2x_1 x_3^2 - x_1 x_2 x_4 - x_2^2 x_3 = \frac{1}{6}\frac{\partial f}{\partial x_3},$$

$$X_3 = x_1 x_3 x_4 - 2x_2^2 x_4 + x_2 x_3^2 = -\frac{1}{6}\frac{\partial f}{\partial x_2},$$

$$X_4 = x_1 x_4^2 - 3x_2 x_3 x_4 + 2x_3^3 = \frac{1}{2}\frac{\partial f}{\partial x_1},$$

it may be verified that $f(X_1, X_2, X_3, X_4) = f^3(x_1, x_2, x_3, x_4)$. This follows at once if it be noticed that $f(X)$ is the discriminant of the cubicovariant of the basic binary cubic, and further that all the invariants of the binary cubic are expressible in terms of the discriminant.

Cayley has given a different proof and at the same time a generalization|| of Eisenstein's result. If we set

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‡ This Bulletin, vol. 33 (1927), pp. 71–80.

§ Journal für Mathematik, vol. 27 (1844), p. 105.

|| A. Cayley, Collected Mathematical Papers, vol. 1, 1889, No. 15, *Note sur deux formules données par MM. Eisenstein et Hesse*, pp. 113–116.

$$u(x) = x_1^2 x_8^2 + x_2^2 x_7^2 + x_3^2 x_6^2 + x_4^2 x_5^2 + 4x_1x_4x_6x_7 + 4x_2x_3x_5x_8$$
$$- 2x_1x_2x_7x_8 - 2x_1x_3x_6x_8 - 2x_1x_4x_5x_8$$
$$- 2x_2x_3x_6x_7 - 2x_2x_4x_5x_7 - 2x_3x_4x_5x_6,$$
$$X_i = \frac{1}{2}\frac{\partial u}{\partial x_i}, \qquad (i = 1, \cdots, 8),$$

then Cayley's result is $u(X) = u^3(x)$. In another paper,* Cayley proves that the Hessian of Eisenstein's quaternary quartic, that is, $f(x_1, x_2, x_3, x_4)$ above, is the product of a numerical constant into the square of f. However, no connection between this property and the automorphic property is indicated.

Another instance of (1) is furnished by the general symmetric determinant of order n: its adjoint is also symmetric and is equal to the $(n-1)$ st power of the original determinant. This is of course true of the general determinant. However the latter is quite trivial since the general determinant, regarded as a function of n^2 independent variables, is *composable*, and clearly all composable forms have the automorphic property (1). It should be noticed that here again the automorphic transformation is expressed in terms of partial derivatives of the first order.

In this note we consider diophantine automorphisms characterized by this property. We assume a form f of degree k in n variables such that

$$(2) \qquad \begin{cases} f(X_1, \cdots, X_n) = \alpha f^{k-1}(x_1, \cdots, x_n), \\ X_i = \sum_{j=1}^{n} c_{ij}\dfrac{\partial f(x)}{\partial x_j}, \qquad \left|\dfrac{\partial X_i}{\partial x_j}\right| \neq 0, \end{cases}$$

where α and c_{ij} are numerical. We prove two results.

(I) The transformation defined by the second of (2) is a Cremona transformation, and (but for a linear transformation) is of period two:

$$(3) \qquad X_i(X) = \alpha f^{k-2} \sum_{j=1}^{n} \gamma_{ij} x_j.$$

This result is a sort of converse to a result of L. Weisner,† who shows that from a Cremona transformation of finite period a diophantine automorph may be deduced.

* Loc. cit., No. 54, *Note sur les hyperdéterminants*, pp. 352–355.

† This Bulletin, vol. 33 (1927), pp. 707–712.

(II) If $f(x_1, \cdots, x_n)$ is irreducible (in the field of rationals, say), then the Hessian of f is a constant multiplied into a power of f:

$$H[f] = \beta f^{n(k-2)/k}. \tag{4}$$

This second result shows the connection between the Cayley identities quoted above. From (4) we have as an immediate corollary that for a function satisfying (2) the degree, k, is a divisor of $2n$, n being the number of variables.

2. *The Cremona Property.* Let us write

$$f_j = \frac{\partial f(x)}{\partial x_j}, \quad Y_i(x) = \sum_{j=1}^{n} c_{ji} f_j.$$

Then since

$$\frac{\partial}{\partial f_i} = \sum_{j=1}^{n} \frac{\partial X_j}{\partial f_i} \frac{\partial}{\partial X_j} = \sum_{j=1}^{n} c_{ji} \frac{\partial}{\partial X_j},$$

we find that

$$\begin{aligned} Y_i(X) &= \sum_j c_{ji} \frac{\partial f(X)}{\partial X_j} = \frac{\partial f(X)}{\partial f_i} = \alpha \frac{\partial}{\partial f_i} f^{k-1}(x) \\ &= \alpha \sum \frac{\partial f^{k-1}}{\partial x_j} \frac{\partial x_j}{\partial f_i} = \alpha(k-1) f^{k-2} \sum f_j \frac{\partial x_j}{\partial f_i} \cdot \end{aligned} \tag{5}$$

But

$$\sum x_j f_j = kf,$$

so that

$$x_i + \sum f_j \frac{\partial x_j}{\partial f_i} = k \frac{\partial f}{\partial f_i} = k \sum \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial f_i} = k \sum f_j \frac{\partial x_j}{\partial f_i},$$

and

$$x_i = (k-1) \sum f_j \frac{\partial x_j}{\partial f_i} \cdot$$

Substituting into (5), we find

$$Y_i(X) = \alpha f^{k-2} \cdot x_i,$$

and

$$X_i(X) = \alpha f^{k-2} \sum \gamma_{ij} x_j, \tag{3}$$

where $\|\gamma_{ij}\| = \|c_{ij}\|^{-1} \cdot \|c_{ij}\|$.

3. *Proof of* (4). It is evident that

$$\left|\frac{\partial Y(X)}{\partial X}\right|\cdot\left|\frac{\partial X}{\partial x}\right| = \left|\frac{\partial Y(X)}{\partial x}\right|.$$

From this fact, and from the equations

$$\left|\frac{\partial Y(X)}{\partial X}\right| = \left| c_{ji}\right| \cdot \left|\frac{\partial^2 f(X)}{\partial X_i \partial X_j}\right| = cH[f(X)],$$

$$\left|\frac{\partial X}{\partial x}\right| = \left| c_{ij}\right| \cdot \left|\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right| = cH[f(x)],$$

where H denotes the Hessian constructed with respect to the variables indicated, we find

$$c^2 H[f(X)]H[f(x)] = \left|\frac{\partial Y(X)}{\partial x}\right| = \alpha^n \left|\frac{\partial (x_i f^{k-2})}{\partial x_j}\right|.$$

The evaluation of this determinant presents no difficulty. Let us use the notation $\phi_j = \partial\phi/\partial x_j$, and let δ_{ij} denote the Kronecker delta. Then we may write

$$\left|\frac{\partial (x_i\phi)}{\partial x_j}\right| \left| = x_i\phi_j + \phi\delta_{ij}\right|$$

$$= -\begin{vmatrix} x_1\phi_1+\phi & x_1\phi_2 & \cdots & x_1\phi_n & x_1 \\ x_2\phi_1 & x_2\phi_2+\phi & \cdots & x_2\phi_n & x_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n\phi_1 & x_n\phi_2 & \cdots & x_n\phi_n+\phi & x_n \\ 0 & 0 & \cdots & 0 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} \phi & 0 & \cdots & 0 & x_1 \\ 0 & \phi & \cdots & 0 & x_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \phi & x_n \\ \phi_1 & \phi_2 & \cdots & \phi_n & -1 \end{vmatrix} = -\{1, 2, \cdots, n\}, \text{ say.}$$

Then we shall have

$$\begin{aligned}\{1, 2, \cdots, n\} &= \phi\{2, \cdots, n\} - x_1\phi_1\phi^{n-1}\\ &= \phi^2\{3, \cdots, n\} - (x_1\phi_1 + x_2\phi_2)\phi^{n-1}\\ &= -\phi^n - (x_1\phi_1 + \cdots + x_n\phi_n)\phi^{n-1} = -(s+1)\phi^n\end{aligned}$$

if ϕ be homogeneous of degree s. Therefore

$$\left|\frac{\partial(x_i f^{k-2})}{\partial x_j}\right| = (k-1)^2 f^{n(k-2)},$$

and substituting in (6),

$$c^2 H[f(X)]H[f(x)] = \alpha^n (k-1)^2 f^{n(k-2)}.$$

If now we make use of the hypothesis that $f(x_1, \cdots, x_n)$ is irreducible we have immediately

$$H[f(x)] = \beta f^{n(k-2)/k}, \tag{4}$$

the exponent being found by comparing the degrees of both members.

4. *Forms Not Satisfying* (2). It is by no means necessary that a form f having the automorphic property satisfy (2). This is obvious if f be composable and of degree $\neq 3$. Furthermore simple examples exist of forms which are not composable. As an example we remark that

$$(x_1^2 + x_2^2 + x_3^2)^2 = (x_1^2 + x_2^2 - x_3^2)^2 + (2x_1x_3)^2 + (2x_2x_3)^2,$$

but, as is well known, $x_1^2 + x_2^2 + x_3^2$ is not composable.

In another note I shall indicate the construction of a class of forms satisfying (1) but in general not (2); the forms are derived from invariant theory.

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