ON REAL SYMMETRIC DETERMINANTS WHOSE PRINCIPAL DIAGONAL ELEMENTS ARE ZERO

BY W. V. PARKER

A paper by L. M. Blumenthal in this Bulletin* is concerned with the following theorem.

If the symmetric determinant

$$D = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} \\ 1 & r_{21} & 0 & r_{23} & r_{24} \\ 1 & r_{31} & r_{32} & 0 & r_{34} \\ 1 & r_{41} & r_{42} & r_{43} & 0 \end{vmatrix}, \qquad (r_{ij} = r_{ji}),$$

with $r_{ij}>0$, $(i,\ j=1,2,3,4)$, $i\neq j$, *is different from zero, and the complementary minors of four of the elements in the principal diagonal vanish, then the complementary minor of the remaining element does not vanish.*

The purpose of this note is to establish a more general theorem from which the above is immediately obtainable. This theorem may be stated as follows.

THEOREM 1. *If the real symmetric determinant,* $D=|a_{ij}|$, *of order five with* $a_{ii}=0$, $(i,\ j=1,2,3,4,5)$, *has all five of its fourth-order principal minors zero, then D vanishes.*

Since the elements of D are real, we may write

$$D = \begin{vmatrix} 0 & a^2 & b^2 & c^2 & d^2 \\ a^2 & 0 & x^2 & y^2 & z^2 \\ b^2 & x^2 & 0 & u^2 & v^2 \\ c^2 & y^2 & u^2 & 0 & w^2 \\ d^2 & z^2 & v^2 & w^2 & 0 \end{vmatrix},$$

where a,b,c,d,x,y,z,u,v,w are either real or pure imaginary numbers. The theorem is trivial if all elements of the first row are zero. We will suppose then that at least one element is different from zero and without loss of generality we may assume $a\neq 0$.

* This Bulletin, vol. 37 (1931), pp. 752–758.

If now we subtract b^2/a^2 times the second column from the third, c^2/a^2 times the second column from the fourth, and d^2/a^2 times the second column from the fifth, and then transform the rows in the same way, D assumes the form

$$(1)\ D = -\frac{1}{a^2}\begin{vmatrix} -2b^2x^2 & a^2u^2-b^2y^2-c^2x^2 & a^2v^2-d^2x^2-b^2z^2 \\ a^2u^2-b^2y^2-c^2x^2 & -2c^2y^2 & a^2w^2-d^2y^2-c^2z^2 \\ a^2v^2-d^2x^2-b^2z^2 & a^2w^2-d^2y^2-c^2z^2 & -2d^2z^2 \end{vmatrix}.$$

If we denote the fourth-order principal minors by M_1, M_2, M_3, M_4, M_5, we have

$$\begin{aligned} M_1 &= (xw+yv+zu)(xw+yv-zu)(xw-yv+zu)(xw-yv-zu), \\ M_2 &= (bw+du+cv)(bw+du-cv)(bw-du+cv)(bw-du-cv), \\ M_3 &= (aw+dy+cz)(aw+dy-cz)(aw-dy+cz)(aw-dy-cz), \\ M_4 &= (av+dx+bz)(av+dx-bz)(av-dx+bz)(av-dx-bz), \\ M_5 &= (au+cx+by)(au+cx-by)(au-cx+by)(au-cx-by). \end{aligned}$$

If $M_3=M_4=M_5=0$, then $a^2u^2=(cx\pm by)^2$, $a^2v^2=(dx\pm bz)^2$, and $a^2w^2=(dy\pm cz)^2$. Substituting these values in (1) we get

$$\begin{aligned} (2)\qquad D &= -\frac{(8b^2c^2d^2x^2y^2z^2)}{a^2}\begin{vmatrix} -1 & \pm 1_u & \pm 1_v \\ \pm 1_u & -1 & \pm 1_w \\ \pm 1_v & \pm 1_w & -1 \end{vmatrix} \\ &= 2[(\pm 1_u)(\pm 1_v)(\pm 1_w)+1](-8b^2c^2d^2x^2y^2z^2)/a^2, \end{aligned}$$

where $\pm 1_u$ is $+1$ or -1 according as $a^2u^2=(cx+by)^2$ or $a^2u^2=(cx-by)^2$ and similarly for $\pm 1_v$ and $\pm 1_w$. From this form it is evident that if b,c,d,x,y, or z is zero, D is zero and hence our theorem is true for this case. Let as assume now that b,c,d,x,y,z are all different from zero. Then in order that D be different from zero it is necessary that the above signs be taken *all plus* or *one plus and two minus*. Since M_1 and M_2 are even functions of u,v,w, we may take only positive square roots on both sides.

CASE I. *The three signs plus.* In this case $au=cx+by$, $av=dx+bz$, $aw=dy+cz$. Substituting these values in the expressions for M_1 and M_2 above we have

$$\begin{aligned} M_1 &= -(dxy+cxz+byz)(16bcdx^2y^2z^2)/a^4, \\ M_2 &= -(cdx+bdy+bcz)(16b^2c^2d^2xyz)/a^4. \end{aligned}$$

Since by supposition b,c,d,x,y,z are all different from zero, in order that $M_1 = M_2 = 0$ it is necessary that $dxy + cxz + byz = 0$ and $cdx + bdy + bcz = 0$, from which it follows that $c^2x^2 + bcxy + b^2y^2 = 0$. That is $cx/(by)$ must be a complex cube root of unity, but this is impossible since $cx/(by)$ is either real or a pure imaginary number. Hence the theorem is true for this case.

CASE II. *One sign plus and two minus.* There are three possibilities here. We will carry the three through simultaneously indicating the corresponding steps by (1), (2), (3) for each case:

$$(1) \begin{cases} au = cx + by, \\ av = dx - bz, \\ aw = dy - cz, \end{cases} \quad (2) \begin{cases} au = cx - by, \\ av = dx + bz, \\ aw = dy - cz, \end{cases} \quad (3) \begin{cases} au = cx - by, \\ av = dx - bz, \\ aw = dy + cz. \end{cases}$$

The corresponding expressions for M_1 and M_2 are

$$(1) \qquad M_1 = -(dxy - cxz - byz)(16bcdx^2y^2z^2)/a^4,$$

$$(2) \qquad M_1 = -(cxz - dxy - byz)(16bcdx^2y^2z^2)/a^4,$$

$$(3) \qquad M_1 = -(byz - dxy - cxz)(16bcdx^2y^2z^2)/a^4,$$

$$(1) \qquad M_2 = -(bcz - bdy - cdx)(16b^2c^2d^2xyz)/a^4,$$

$$(2) \qquad M_2 = -(bdy - bcz - cdx)(16b^2c^2d^2xyz)/a^4,$$

$$(3) \qquad M_2 = -(cdx - bcz - bdy)(16b^2c^2d^2xyz)/a^4.$$

Since b,c,d,x,y,z are all different from zero, in order that $M_1 = M_2 = 0$ it is necessary that

$$(1) \qquad c^2x^2 + bcxy + b^2y^2 = 0,$$

$$(2) \qquad d^2x^2 + bdxz + b^2z^2 = 0,$$

$$(3) \qquad d^2y^2 + cdyz + c^2z^2 = 0.$$

None of these is possible for the same reason as in Case I.

We have shown therefore that if $M_3 = M_4 = M_5 = 0$ and $D \neq 0$, then not both M_1 and M_2 are zero. Hence, if $M_1 = M_2 = M_3 = M_4 = M_5 = 0$, D vanishes.

If the elements outside the principal diagonal of D are all different from zero, we see from (2) that if $M_3 = M_4 = M_5 = 0$ then the signs must be taken *all minus* or *two plus and one minus* in order for D to vanish. In each of these cases $M_1 = M_2 = 0$. We may state, in this case, therefore the following corollary.

COROLLARY. *If four of the principal minors of D are zero and the fifth is not zero then D is not zero.*

As an immediate consequence of Theorem 1 we have also the following result.

THEOREM 2. *If $D = |a_{ij}|$ is a real symmetric determinant of order greater than four with $a_{ii} = 0$, then if all fourth-order principal minors of D are zero, D vanishes.*

Any fifth-order principal minor of D is of the form of the determinant under Theorem 1 and hence is zero if all fourth-order principal minors vanish. Hence all principal minors of orders four and five are zero and therefore the rank of D is three or less.*

THEOREM 3. *If $D = |a_{ij}|$ is a real symmetric determinant of order n, $n > 5$, with $a_{ii} = 0$, and M is any principal minor of D of order $n-1$, then if all fourth-order principal minors of M are zero, D vanishes.*

By Theorem 2, the rank of M is three or less. We may suppose now that M is the minor in the upper left hand corner of D. When D is expanded according to the product of the elements of its last row and last column it is expressed as a sum of products of these elements by minors of M of order $n-2$†. But all minors of M of order $n-2$ are zero and hence D vanishes.

MISSISSIPPI WOMAN'S COLLEGE

* Bôcher, *Introduction to Higher Algebra*, p. 57, Theorem 2.

† Bôcher, loc. cit., p. 29, Theorem 3.