TWO BOOKS ON DIFFERENTIAL GEOMETRY

Eléments de Géométrie Infinitésimale. By Gaston Julia. Paris, Gauthier-Villars, 1927. vi+242 pp.

Lehrbuch der Differentialgeometrie. By A. Duschek and W. Mayer. Band I, *Kurven und Flächen im Euklidischen Raum*, von A. Duschek. vi+250 pp. Band II, *Riemannsche Geometrie*, von W. Mayer. vi+245 pp. Leipzig and Berlin, B. G. Teubner, 1930.

It is interesting to compare Julia's book with the first volume of Duschek-Mayer's work, as they deal with the same subject, classical differential geometry. Both books use invariant notation; both endeavor to be accurate in the formulation of the theorems; both are of about one size. Still they differ greatly in general aspect and in material.

Julia's book follows more the classical pattern, and refers often to the treatises of Picard, Goursat, de la Vallée Poussin, Darboux, and to the work of Humbert. It uses vector methods for the formulation of certain general theorems, but often slips into coordinate notation, as required in special problems.

Duschek's book, on the contrary, persistently attempts to use not only vector methods, but also tensor calculus in ordinary differential geometry. As such, it is a pioneer work, with the possible exception of Ricci's never printed *Lezioni sulla Teoria delle Superficie* and J. E. Campbell's *Course of Differential Geometry* (1926), which, however, differ very much from Duschek's treatise. This results in a tendency to dwell upon theorems of a general nature.

A striking difference lies in the large amount of space that Julia devotes to the theory of contact and to the theory of envelopes (pp. 9–72, one-fourth of the book). Duschek devotes to this subject only a short discussion. Is it because Julia is in first instance an analyst? The modern theory of contact and envelopes was indeed introduced into differential geometry by two analysts, Lagrange and Cauchy. This fact makes Julia's book one of the best sources of information on contact and on envelopes. We find here contact of plane curves, of space curves, of curves and surfaces, and of surfaces. Then we have the discussion of the envelopes of systems of

(a) plane curves of equation $f(x, y, \alpha)=0$;
(b) plane curves of equation $f(x, y, \alpha, \beta)=0, \ \phi(\alpha, \beta)=0$;
(c) surfaces of equation $f(x, y, z, \alpha)=0$;
(d) surfaces of equation $f(x, y, z, \alpha, \beta)=0, \ \phi(\alpha, \beta)=0$;
(e) surfaces of equation $f(x, y, z, \alpha, \beta)=0$;
(f) space curves of equation $f(x, y, z, \alpha)=0, \ g(x, y, z, \alpha)=0$;
(g) space curves of equation $f(x, y, z, \alpha, \beta)=0, \ g(x, y, z, \alpha, \beta)=0$;

which discussion leads to the theory of congruences of curves, with mention of focal properties.

We should have liked to see here a more detailed discussion of the behavior of the envelope, taking higher derivatives into account. Apart from the paper by Risley and McDonald in the Annals of Mathematics of 1910–11 (second series, vol. 12), and a paper by de la Vallée Poussin, reprinted in the sixth edition of his *Cours d'Analyse Infinitésimal* (Tome II, 1928, published after Julia's book), very little has been done on this important subject.

In later parts of the book Julia remains interested in the subject of congruences. He studies line congruences, especially normal congruences, with some interesting examples. Ruled surfaces and developable surfaces are discussed.

Other points treated in Julia's book are the classical theory of space curves and surfaces, Bour's theorem, and conformal representation applied to map projection. In an appendix we find remarks on imaginary elements and on vector notation. Vector notation is, in this treatise, just a short way to write three component equations in euclidian three-space. It is not used for curvilinear coordinates on the surface.

Duschek's book, on the contrary, has the use of Ricci notation as its special feature. The selection of the material is in accordance with this notation. In an introduction he explains the principal group conceptions and the Erlanger program. Vector notation is replaced by component notation v_i, $i=1, 2, 3$, in accordance with the dialectic method of Ricci calculus, which brings vector calculus to a higher perfection by the consistent use of components. Scalar products are written v_iw_i, vector products are obtained with the aid of the unit trivector e_{ijk}, so that

$$e_{ijk}v_jw_k = p_i$$

stands for

$$\mathbf{v} \times \mathbf{w} = \mathbf{p}$$

in the Gibbs notation.

The elementary theory of curves can easily be translated into such notation. For the surface a new set of vectors and tensors is necessary, those defined with respect to transformations of curvilinear coordinates on the surface. Here we get vectors v_α, $\alpha=1, 2$, and tensors $p_{\alpha\beta}$, $\alpha, \beta=1, 2$. Latin indices run from 1 to 3, Greek indices from 1 to 2. Now we can develop readily the general properties of curves on the surface. We also find a discussion of analytic curves with complex variables, of so-called ametric parameters on analytic surfaces in the complex domain (these are parameters for which $ds^2=g_{12}du_1du_2$), of the determination of a surface by its first and second differential form, of Gauss-Bonnet's theorem, of geodesic conic sections, and of Liouville surfaces. As an example of differential geometry in the large we find the theorem on the impossibility of deforming a closed ovaloid.

Julia and Duschek differ in the way they introduce the positive direction of the principal normal to a space curve. For Julia this direction is that from the point of the curve to the center of curvature. The radius of curvature is always positive. Duschek, as does Blaschke, leaves the direction indeterminate, the curvature k and the unit normal vector $\mathbf{n}$ being only related by the formula $d\mathbf{t}/ds=k\mathbf{n}$, $\mathbf{t}$ representing the unit tangent vector. Julia's method leads to a difficulty in the case of plane curves, where a point of inflexion suddenly turns the moving "two-leg" through an angle of 180°. He therefore changes the definition for the plane. Duschek's definition does not lead to this, and it can be shown that in the plane the sign of k can be uniquely defined.

In Duschek's book there is no thorough discussion of the relations between tensors on the surface and tensors in space. The "induction" of a tensor in a surface by a tensor in space is not introduced. This is, however, regularly done in the second volume of Duschek-Mayer's treatise. In the first volume we therefore find the second fundamental tensor $b_{\alpha\beta}$ defined in this way:

$$b_{\alpha\beta} = \frac{\partial^2 x_i}{\partial u_\alpha \partial u_\beta} \nu^i, \qquad (i = 1, 2, 3; \alpha, \beta = 1, 2),$$

where ν^i are the components of the surface normal and $x_i = x_i(u_\alpha)$ are the equations of the surface. This is sufficient for ordinary differential geometry.

Covariant differentiation on the surface is introduced after the discussion of space curves and the elementary theory of surfaces. It is mainly used for a renewed discussion of the Gaussian curvature tensor, for the proof that a surface is determined in shape by its two fundamental tensors, for infinitesimal deformation, and for geodesics.

A chapter on some special subjects (ruled surfaces, minimal surfaces) concludes this interesting book.

Walther Mayer has written the second volume of Duschek-Mayer's textbook on differential geometry, and it has in many respects a character similar to that of the first volume. It deals with Riemannian geometry, and is a very valuable addition to the literature on this subject. It contains a careful introduction to tensor calculus, without which the study of Riemannian manifolds seems to be very difficult. Then follow a chapter on curves in euclidean n-space with the Frenet formulas for Riemannian manifolds, and a forty-page introduction to the calculus of variations with application to geodesics. After this we get the parallel displacement of Levi-Civita, geodesic manifolds, and a chapter on manifolds immersed in a Riemannian manifold. The spaces of constant Riemannian curvature receive special attention. A special feature of the book is Chapter IX, containing what the author calls "Das Formenproblem." Here an old question is finally settled, namely the generalization to n dimensions of Bonnet's theorem that a surface is determined in shape by its first and second fundamental tensors. Here the complete theory is given for an l-dimensional manifold ($l < n$), in an n-dimensional euclidean space. The formulas constructed for this purpose are the generalization of the Frenet formulas to this general case. Since that time Schouten and van Kampen have restated the results in another way (Mathematische Annalen, vol. 105 (1931); see the remark by Duschek in Zentralblatt für Mathematik, 26 Oktober 1931, p. 153).

An appendix gives a generalization of Meusnier's theorem, the integral theorem of Gauss in Riemannian manifolds, and the derivation of the Eulerian equations for the gyroscope in tensor calculus, where they appear as an almost trivial result of the fundamental equations.

As a result of the collaboration of the two Viennese scholars we therefore have an extremely interesting addition to our textbooks on differential geometry.

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