VAN DER WAERDEN ON ALGEBRA

Moderne Algebra. By B. L. van der Waerden. Unter Benutzung von Vorlesungen von E. Artin und E. Noether. Berlin, Springer, 1931. Vol. I. viii+243 pp. Vol. II. vii+216 pp.

It will immediately occur to every reader of van der Waerden's new book that modern algebra is a subject quite different from the classical algebra built up in its last golden era by Dedekind, Weber, Frobenius, and Kronecker. It is true that a closer study often reveals that the main difference lies in the form of presentation, but it is equally true that in many instances the problems of modern algebra are broader and of a different character.

The new school of abstract algebra has developed into one of the strongest branches of present day mathematics in Germany. Its fundamental principles are closely related to Hilbert's ideas of a formal foundation of mathematics, reducing all theories to an axiomatic basis consisting of relational properties of undefined elements. It is of course nothing new to build up a mathematical theory from its axioms; the main problem of abstract algebra is however the determination of *all* systems with a given operational basis, i.e. to find the structural properties of all such systems. It is interesting to observe to what a remarkable extent this has been possible. An immediate consequence is an intimate knowledge of the fundamental assumptions of each theory and theorem; but still more important is the abstract identification of many mathematical investigations, also outside of algebra proper, which makes abstract algebra a unifying principle sorely needed in these times of specialization.

One of the central papers in abstract algebra is the well known analysis by Steinitz of the structure of fields. It has been followed by a vast number of investigations on the structure of groups, rings, ideals, hypercomplex systems, etc., works associated with the names of Artin, Krull, E. Noether, and others. For hypercomplex systems the investigations of Wedderburn and Dickson are outstanding.

This new *Algebra* proposes to be a guide to these investigations, and in many ways it is more than that; van der Waerden has coordinated the various investigations and he has tried as far as possible to consider them from the most general point of view. His book therefore gives considerably more than a summary of the previous theory, and it will certainly keep a prominent place among the books on algebra for many years to come. It is not a text book in the ordinary sense, and it requires a wide preliminary knowledge on the part of the reader, even though every subject is worked up from its foundations. But I am certain that advanced students and mathematicians interested in algebra will study it with great pleasure.

The book is clearly written, well balanced in its content, and there is much to be praised and little to be criticised. In places the desire for great generality has made it a little obscure; the problems also seem somewhat too simple and I should have liked to have a more complete list of references. Due to the large quantity of facts which had to be crowded into a short book, one sometimes feels that one would have preferred to have a little more of every subject, but this is after all only the earmark of a good book.

The first volume is on the whole more elementary than the second, and is in itself a good introduction to the abstract theory of fields, groups, and ideals. It starts with a discussion of numbers, sets, and the foundations of group theory. The next chapter contains the fundamental properties of rings and fields, isomorphisms, properties of polynomials in a ring, the principal concepts of the abstract ideal theory, and a discussion of the decomposition into prime elements. After constructing a quotient field for a commutative ring, van der Waerden observes that the corresponding problem for non-commutative rings is still unsolved; in a recent paper I have shown that in the case of a non-commutative ring without divisors of zero, the set of all formal quotients will only form a field in case the ring has the property that for any two given elements a and b one can find two others $a_1 \neq 0$, $b_1 \neq 0$ such that $a_1 a = b_1 b$. There are, however, as I have indicated by an example, other methods of enlarging a ring to a field, but it seems very hard to find a general principle for such a construction.

In the fourth chapter one finds the theory of polynomials and symmetric functions in a ring R or a field F. The decomposition into prime polynomials is treated at some length; among other results one will find the following: If the elements of the ring R are factorable in a finite number of steps, and R has only a finite number of units, then all polynomials with coefficients in R are factorable in a finite number of steps. The proof follows by a method of Kronecker. I should like to point out that one can give a necessary and sufficient condition from which the criterion of van der Waerden immediately follows, viz: The factors of polynomials with coefficients in a field F can be found in a finite number of steps, if the first degree factors can be determined in this way. It is only necessary to consider the case where the polynomial has different roots; if it has a factor of mth degree, one constructs the equation in F satisfied by any symmetric function of m of the roots; the possible roots in F of this equation give the possible coefficients of a factor.

The long Chapter 5 gives the abstract theory of fields in a somewhat simpler form than does Steinitz; it should be observed that van der Waerden calls an irreducible polynomial *separable* if all of its roots are different, *inseparable* if this is not the case; these terms are more suggestive than the *first* and *second kind* of Steinitz. In Chapter 7 the author returns to group theory and also treats the groups with operators studied by Krull and Noether, and then follow the applications to the general Galois theory of (finite) algebraic extensions. The fundamental correspondence-theorem for subgroups and subfields holds for all separable extensions and only for these. It is to be noted, however, that when the characteristic of the field is not zero, the Galois criterion for solvability by radicals does not apply to all cases.

In the following chapter one finds the proof of the existence of algebraically closed fields and the invariance of the degree of transcendency; these parts of the theory were rather complicated in the original treatment by Steinitz, but van der Waerden has succeeded in giving a clear-cut presentation. The last chapter contains Artin and Schreier's theory of abstract real fields and also a short paragraph on absolute values (Bewertungen). It would have been interesting to have a more complete discussion of the latter theory, which contains many interesting results, but has never been presented in any textbook.

The second volume deals mainly with general ideal theory and non-commutative algebra. It would be tempting to discuss at some length the several new subjects which are here treated for the first time in a textbook, but I shall have to limit myself to a short description of the main character of the chapters.

Chapter 11 contains the general elimination theory of functions of several variables. The existence of a system of resultants is demonstrated, and used to prove the fundamental theorem of Hilbert. As an application the theorem of Bezout is proved for an arbitrary number of variables. Chapter 12 contains commutative ideal theory in domains in which E. Noether's chain condition holds; the principal result is the theorem by Lasker on the representation of an ideal as the greatest common factor of primary components. In Chapter 13 this theory is applied to the theory of polynomial ideals, which is virtually algebraic geometry in abstract form. One finds among other topics a rather complete discussion of the fundamental theorem of M. Noether, deduced from the theorem of Lasker. This abstract algebraic geometry has lately been one of the favorite subjects of the author, and I should have liked to see some of his further investigations included, for instance, on the "Abzählungsprinzip" of Zeuthen.

Chapter 14 contains investigations of Noether and Krull on the ideal theory in maximal integral (ganzabgeschlossenen) rings; this theory includes the foundation of the ordinary ideal theory in algebraic fields and it is directly inspired by the theory of Dedekind. Chapter 15 deals with linear systems, moduli, matrices, and elementary divisors for commutative and non-commutative domains. One finds the main properties of the generalized Abelian groups studied by Krull, and at the end a paragraph on quadratic and Hermitian forms. Chapter 16 contains the principal theorems on the structure of hypercomplex systems; the presentation is short, too short I am afraid, for a reader who is not already somewhat familiar with the subject. The final chapter contains the application of this theory to the representation of groups and the theory of characters of finite groups.

OYSTEIN ORE