THE INDEPENDENCE OF THE POSTULATES OF LOGIC

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Though more than twenty years have elapsed since the publication of the first edition of *Principia Mathematica*, no proofs of the independence of its primitive propositions have appeared. Such proofs are given here as holding both for the postulates in their original logistic form, and for Bernstein's mathematical transcription.†

The postulates whose independence is to be proved are, in their logistic form, as follows:

*1·1 Anything implied by a true elementary proposition is true.

*1·2 $\vdash : p \vee p . \supset . p.$

*1·3 $\vdash : q . \supset . p \vee q.$

*1·4 $\vdash : p \vee q . \supset . q \vee p.$

*1·6 $\vdash :. q \supset r . \supset : p \vee q . \supset . p \vee r.$

*1·7 If p is an elementary proposition, $\sim p$ is an elementary proposition.

*1·71 If p and q are elementary propositions, $p \vee q$ is an elementary proposition.

Or, in Bernstein's "mathematicized" form:

1.1 There exists a K-element 1, such that from $p=1$ and $p'+q=1$ follows $q=1$.‡

1.2 $(p+p)'+p=1.$

1.3 $q'+(p+q)=1.$

1.4 $(p+q)'+(q+p)=1.$

1.6 $(q'+r)'+[(p+q)'+(p+r)]=1.$

1.7 If p is a K-element, p' is a K-element.

1.71 If p and q are K-elements, $p+q$ is a K-element.

† B. A. Bernstein, *Whitehead and Russell's theory of deduction as a mathematical science*, this Bulletin, vol. 37 (1931), p. 480.

‡ This postulate is not an accurate transcription of *1.1 unless the convention be adopted that the postulate is not satisfied by any case in which the hypothesis is not satisfied. See *Principia Mathematica*, p. 110. This convention will be assumed throughout.

Three primitive propositions which appear in the first edition of the *Principia* do not appear in these lists. These are $*1\cdot 11$ which was found unnecessary in the second edition,† $*1\cdot 5$ which was proved redundant by Bernays,‡ and $*1\cdot 72$ which, not being concerned with elementary propositions, does not interest us here.

The independence proofs follow.

EXAMPLE 1.1.§ K = a class of two elements, 1, 2.

+	1	2	′
1	1	1	2
2	1	1	1

When $p=1$ and $p'+q=1$, q may equal 2; hence postulate 1.1 fails. All other postulates are satisfied.

EXAMPLE 1.2.‖ K = a class of three elements, 1, 2, 3.

+	1	2	3	′
1	1	1	1	3
2	1	1	2	2
3	1	2	3	1

When $p=2$, $p+p=1$, $(p+p)'=3$, and $(p+p)'+p=2$. Hence 1.2 fails; but all other postulates hold.

EXAMPLE 1.3. K = a class of four elements, 1, 2, 3, 4.

+	1	2	3	4	′
1	1	1	1	1	4
2	1	4	1	4	3
3	1	1	4	4	2
4	1	4	4	4	1

† See p. xiii, second edition.

‡ P. Bernays, *Axiomatische Untersuchung des Aussagen-Kalkuls der "Principia Mathematica,"* Mathematische Zeitschrift. vol. 25 (1926), p. 305.

§ In these examples, the values for the variable p are given in the vertical left hand margin of the table, and corresponding values for p' (or $\sim p$) are given in the column at the extreme right. The values for q are given in the upper horizontal margin, and the values of $p+q$ (or $p \vee q$) are given in the square. Where p has value x and q has value y, the value of $p+q$ is to be found on the same horizontal line as x and the same vertical line as y.

In every case $\vdash \cdot p$ shall be interpreted $p=1$, and the values of $p \supset q$ may be determined by the definition $p \supset q . = . \sim p \vee q$.

‖ This example is due to Bernays, loc. cit.

When, for example, $p=q=2$, $q'=3$, and $p+q=4$. Hence $q'+(p+q)=3+4=4$ and postulate 1.3 fails. All others hold.

EXAMPLE 1.4. K = a class of four elements, 1, 2, 3, 4.

+	1	2	3	4	′
1	1	1	1	1	4
2	1	2	1	2	3
3	1	1	3	4	2
4	1	2	3	4	1

When $p=4$ and $q=3$, $p+q=3$ and $(p+q)'=2$. $q+p=4$, however, so $(p+q)'+(q+p)=2+4=2$. Hence 1.4 fails. All other postulates hold.

EXAMPLE 1.6. K = a class of six elements.

+	1	2	3	4	5	6	′
1	1	1	1	1	1	1	6
2	1	2	1	1	1	2	5
3	1	1	3	1	3	3	4
4	1	1	1	4	4	4	3
5	1	1	3	4	5	5	2
6	1	2	3	4	5	6	1

When, for example, $p=2$, $q=4$, $r=2$; $q'+r=1$, $p+q=1$, and $p+r=2$. $(p+q)'+(p+r)=6+2=2$. $(q'+r)'+[(p+q)'+(p+r)]=6+2=2$. Hence 1.6 fails. All other postulates hold.

EXAMPLE 1.7. K = the subclass of elements, 1, 2, of the four, 1, 2, 3, 4.

+	1	2	3	4	′
1	1	1	1	1	4
2	1	2	1	2	3
3	1	1	3	3	2
4	1	2	3	4	1

Here $1'$ and $2'$ are not members of K; but all other postulates hold.

EXAMPLE 1.71. K = the sub-class of elements 1, 2, 4, 5, 7, 8 of the class, 1, 2, 3, 4, 5, 6, 7, 8.

+	1	2	3	4	5	6	7	8	′
1	1	1	1	1	1	1	1	1	8
2	1	2	1	2	1	2	1	2	7
3	1	1	3	3	1	1	3	3	6
4	1	2	3	4	1	2	3	4	5
5	1	1	1	1	5	5	5	5	4
6	1	2	1	2	5	6	5	6	3
7	1	1	3	3	5	5	7	7	2
8	1	2	3	4	5	6	7	8	1

Here $4+7=7+4=3$, where 3 is not a member of K. All other postulates hold.

In order to deduce Boolean algebra, an independent postulate must be added to the above set:†

1.73. If $p'+q=1$ and $q'+p=1$, then $p=q$.
This postulate will be found to hold for all examples given except example 1.1. If we were not interested in the holding of this additional postulate, simpler examples might have been constructed in some cases. The only example on which postulate 1.1 fails without 1.73 or some other postulate failing is the null example, where K is a class of no elements. Here postulate 1.1, being an existence postulate, fails, while all others are satisfied vacuously.

The consistency of the set is shown by the following example.‡

EXAMPLE 1.8. K = a class of two elements 1, 2.

+	1	2	′
1	1	1	2
2	1	2	1

† See a paper by E. V. Huntington in the Proceedings of the National Academy of Sciences, February, 1932.

‡ Consistency may also be established by examples 1.7 and 1.71 using, in each case, the entire class of elements. These examples are important in establishing the fact that *Principia Mathematica* is not limited to a class of two elements, that is, a calculus of two truth-values.

Of the above postulates, 1.7 and 1.71 are relatively unimportant in the manipulation of the system. These will not be considered further; but for the remaining postulates, proof of complete independence in the sense of E. H. Moore will be given. The complete independence of postulates 1.2, 1.3, 1.4, and 1.6 will be proved first, by using examples for which postulate 1.1 holds. Later a rule will be given for the construction of examples on which postulate 1.1 fails.

A table of examples follows, + indicating the holding of the postulate, and − the failing.

Example	Postulate 1.2	1.3	1.4	1.6
1.8	+	+	+	+
1.2	−	+	+	+
1.3	+	−	+	+
1.4	+	+	−	+
1.6	+	+	+	−
1.23	−	−	+	+
1.24	−	+	−	+
1.26	−	+	+	−
1.34	+	−	−	+
1.36	+	−	+	−
1.46	+	+	−	−
1.234	−	−	−	+
1.236	−	−	+	−
1.246	−	+	−	−
1.346	+	−	−	−
1.2346	−	−	−	−

The examples themselves follow without comment.†

† In these examples, the margins indicating the values of p and q have been omitted, as have the values of p'. However, p' may be determined in every case by the following rule. The number assigned to p' shall be such that if it be added to the number assigned to p, their sum shall be one greater than n, where n is the number of elements in the example.

Example 1.23

2	1
1	2

Example 1.24

1	1	1	1
1	1	1	2
1	1	3	4
1	2	3	4

Example 1.26

1	1	1	1
1	1	1	2
1	1	2	3
1	2	3	2

Example 1.34

1	1	1	1
1	2	1	4
1	1	4	4
1	2	4	4

Example 1.36

1	1	1	1
1	2	1	2
1	1	3	4
1	2	4	4

Example 1.46

1	2	1	1
1	2	1	3
1	1	3	1
1	2	3	4

Example 1.234

1	3	1
1	1	3
1	3	3

Example 1.236

3	3	1
3	1	3
1	3	3

Example 1.246

1	2	1	1
1	1	1	4
1	1	3	4
1	2	3	4

Example 1.346

1	1	1	1
1	2	1	2
1	1	3	3
1	2	4	4

Example 1.2346

2	2
1	2

To obtain corresponding examples on which postulate 1.1 fails but the other holdings and failings are unaffected, it is necessary only to add another element k to each of the above examples. The functions of k are determined as follows:

$$\begin{aligned} a+k &= a+1, \\ k+a &= 1+a, \\ k+k &= 1+1, \\ k' &= 1', \end{aligned}$$

where a is any element in the example.

It is obvious that $f(k)=f(1)$, and any function that equals 1 with 1 as argument will equal 1 with k as argument. Similarly any function unequal to 1 with 1 as argument will be unequal to 1 with k as argument. Hence the holding or failing of the above four postulates will be undisturbed by the introduction of k. In the case of postulate 1.1, however, when $p=1$ and $p'+q=1$ it does not follow that $q=1$. But q may equal k since, ex hypothesi, $p'+k=p'+1$.

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