ON THE CONTACT OF TWO SPACE CURVES*

BY E. B. STOUFFER

If two space curves C and C' have contact of order n at a point P, there exists a unique plane, called the *principal plane*, which passes through the common tangent and which has the property that the cones projecting the curves from any point Q of this plane have contact of at least order $n+1$ along the line PQ. This theorem is due to Halphen.†

Bompiani‡ has shown in the general case where the principal plane is distinct from the common osculating plane that the contact of the projecting cones will be of at least order $n+2$ if Q is on a unique line passing through P and lying in the principal plane and of at least order $n+3$ if Q is a unique point on this line. The line and point are called by Bompiani the *principal line* and the *principal point.*

Halphen proved his part of the theorem by cutting the two curves by a plane and finding the limiting position of the line joining the two points of intersection. Bompiani used properties of surfaces having certain orders of contact with the two curves. More recently Palozzi§ has obtained all parts of the theorem by means of the projective invariant of contact.

It is the purpose of the present paper to prove all three parts of the theorem by a single process which is both direct and elementary, and which has the further advantage that it may be easily extended to prove similar theorems in hyperspace.

Let us assume the two curves C and C' have contact of order‖ $n(n>1)$ at a general point P. Let x, y, z represent the non-homogeneous coordinates obtained from the projective homogeneous coordinates x_1, x_2, x_3, x_4 by means of the relations

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† Journal de L'Ecole Polytechnique, vol. 28 (1880), pp. 25–27.

‡ Memorie della Accademia di Bologna, Classe di Scienze Fisiche, (8), vol. 3 (1925–26), pp. 3–6.

§ Rendiconti Lincei, (6), vol. 7 (1928), pp. 321–25. Also Atti del Congresso Internazionale dei Matematici, Bologna, vol. 4, 1928, pp. 385–88.

‖ If $n=1$, the method of this paper will apply but the coefficients of x^2 in the expressions for z in equations (1) and (2) must be retained in all the calculations since the osculating planes at P are not common in this case.

$$x = \frac{x_2}{x_1}, \quad y = \frac{x_3}{x_1}, \quad z = \frac{x_4}{x_1}\cdot$$

If we choose the point of contact to be the origin, the common tangent to be the x-axis, and the common osculating plane to be the xy-plane, the equations of C and C' may be put into the form

$$(1)\quad \begin{aligned} y &= l_2x^2 + \cdots + l_nx^n + l_{n+1}x^{n+1} + l_{n+2}x^{n+2} + l_{n+3}x^{n+3} + \cdots, \\ z &= m_3x^3 + \cdots + m_nx^n + m_{n+1}x^{n+1} + m_{n+2}x^{n+2} + \cdots, \end{aligned}$$

for C and

$$(2)\quad \begin{aligned} y &= l_2x^2 + \cdots + l_nx^n + l'_{n+1}x^{n+1} + l'_{n+2}x^{n+2} + l'_{n+3}x^{n+3} + \cdots, \\ z &= m_3x^3 + \cdots + m_nx^n + m'_{n+1}x^{n+1} + m'_{n+2}x^{n+2} + \cdots, \end{aligned}$$

for C'. Since we are taking P to be a general point on the two curves we shall assume $l_2 \neq 0$. Moreover, at least one of the inequalities $l_{n+1} \neq l'_{n+1}$, $m_{n+1} \neq m'_{n+1}$ must hold if the order of contact is no higher than n. We shall for the present assume $m_{n+1} \neq m'_{n+1}$. The special case in which $m_{n+1} = m'_{n+1}$ will be discussed later.

Let us now make a transformation of coordinates which shall involve merely the changing of the fourth vertex (0, 0, 0, 1) of the homogeneous coordinate system to a point whose coordinates are a, b, c, 1, where a, b, c are constants as yet undetermined. This transformation is expressed in terms of the non-homogeneous coordinates by the equations

$$(3)\qquad X = \frac{x - bz}{1 - az}, \quad Y = \frac{y - cz}{1 - az}, \quad Z = \frac{z}{1 - az}\cdot$$

The substitutions from (1) and (2) into the second equation of (3) give, respectively,

$$(4)\quad \begin{aligned} Y = l_2x^2 + L_3x^3 + \cdots + L_nx^n + L_{n+1}x^{n+1} + L_{n+2}x^{n+2} \\ + L_{n+3}x^{n+3} + \cdots, \end{aligned}$$

$$(5)\quad \begin{aligned} Y = l_2x^2 + L_3x^3 + \cdots + L_nx^n + L'_{n+1}x^{n+1} + L'_{n+2}x^{n+2} \\ + L'_{n+3}x^{n+3} + \cdots. \end{aligned}$$

The identity of the coefficients of corresponding powers of x, less than x^{n+1}, in these two equations is evident from the fact that they are formed in exactly the same manner from a, b, c and those coefficients of (1) and (2) which are identical.

Furthermore, it is easily seen that

$$
\begin{aligned}
L_{n+1} &= l_{n+1} - cm_{n+1} + \cdots, \\
L'_{n+1} &= l'_{n+1} - cm'_{n+1} + \cdots, \\
L_{n+2} &= l_{n+2} - cm_{n+2} + \cdots, \\
L'_{n+2} &= l'_{n+2} - cm'_{n+2} + \cdots, \\
L_{n+3} &= l_{n+3} - cm_{n+3} + al_2m_{n+1} + \cdots, \\
L'_{n+3} &= l'_{n+3} - cm'_{n+3} + al_2m'_{n+1} + \cdots,
\end{aligned}
\tag{6}
$$

where the terms omitted are identical for the corresponding coefficients of (4) and (5).

In order to complete the transformation it is necessary to eliminate x by the introduction of X in equations (4) and (5). The substitutions from (1) and (2) into the first equation of (3) give, respectively,

$$X = x + \alpha_3x^3 + \cdots + \alpha_nx^n + \alpha_{n+1}x^{n+1} + \alpha_{n+2}x^{n+2} + \cdots, \tag{7}$$

$$X = x + \alpha_3x^3 + \cdots + \alpha_nx^n + \alpha'_{n+1}y^{n+1} + \alpha'_{n+2}x^{n+2} + \cdots, \tag{8}$$

where

$$
\begin{aligned}
\alpha_{n+1} &= - bm_{n+1} + \cdots, \\
\alpha'_{n+1} &= - bm'_{n+1} + \cdots, \\
\alpha_{n+2} &= - bm_{n+2} + am_{n+1} + \cdots, \\
\alpha'_{n+2} &= - bm'_{n+2} + am'_{n+1} + \cdots,
\end{aligned}
\tag{9}
$$

the terms omitted in (9) being identical for corresponding coefficients of (7) and (8).

In order to eliminate successive powers of x from (4) it is only necessary to multiply the square of both sides of (7) by the proper factor and subtract from (4), then multiply the cube of both sides of (7) by the proper factor and subtract from the result, and continue this process as long as desired. In the same manner, successive powers of x may be eliminated from (5) by means of (8). The results of these eliminations have the form

$$(10)\qquad Y = \lambda_2 X^2 + \cdots + \lambda_n X^n + \lambda_{n+1} X^{n+1} + \lambda_{n+2} X^{n+2} + \lambda_{n+3} X^{n+3} + \cdots,$$

$$(11)\qquad Y = \lambda_2 X^2 + \cdots + \lambda_n X^n + \lambda'_{n+1} X^{n+1} + \lambda'_{n+2} X^{n+2} + \lambda'_{n+3} X^{n+3} + \cdots.$$

The fact that the coefficients of corresponding powers of X, less than X^{n+1}, are equal is evident since a transformation of coordinates will not change the order of contact. The fact may also be easily seen analytically. Furthermore the analytical process shows at once that

$$(12)\qquad \begin{aligned} \lambda_{n+1} &= L_{n+1} + \cdots, & \lambda'_{n+1} &= L'_{n+1} + \cdots, \\ \lambda_{n+2} &= L_{n+2} - 2l_2\alpha_{n+1} + \cdots, & \lambda'_{n+2} &= L'_{n+2} - 2l_2\alpha'_{n+1} + \cdots, \\ \lambda_{n+3} &= L_{n+3} - 2l_2\alpha_{n+2} - 3(l_3 - cm_3)\alpha_{n+1} + \cdots, \\ \lambda'_{n+3} &= L'_{n+3} - 2l_2\alpha'_{n+2} - 3(l_3 - cm_3)\alpha'_{n+1} + \cdots, \end{aligned}$$

where as before the terms omitted are identical for the corresponding coefficients of (10) and (11).

The three parts of the theorem now follow immediately. If we give c the unique value imposed by putting $\lambda_{n+1} = \lambda'_{n+1}$, the cones projecting C and C' from the fourth vertex of our new coordinate system have contact of at least order $n+1$. Moreover, since a and b are still arbitrary the fourth vertex may be at any point in a unique plane, the *principal plane*, which contains the two vertices (1, 0, 0, 0) and (0, 1, 0, 0) of our coordinate system, and therefore the tangent line at P to C and C'.

If we give b the unique value imposed by putting also $\lambda_{n+2} = \lambda'_{n+2}$, the projecting cones have contact of at least order $n+2$ and the point of projection is any point on a line through P, the *principal line*.

Finally, if we give a the unique value imposed by putting also $\lambda_{n+3} = \lambda'_{n+3}$, the projecting cones have contact of at least order $n+3$ and the point of projection is a fixed point, the *principal point*.

In the special case where $m_{n+1} = m'_{n+1}$ the above process shows that the fourth vertex cannot be so located as to make the cones projecting C and C' from it have contact of order $n+1$. However, it is evident at once that under this condition the cones projecting C and C' from (0, 0, 1, 0) have contact of at least order $n+1$. Moreover, by changing this vertex to the point $(a, b, 1, 0)$ it is easily shown by a method similar to that used in the general case that the cones projecting C and C' from any point in the osculating plane have contact of order $n+1$. In other words, this special case arises when the principal plane coincides with the osculating plane.

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ON RECTIFIABILITY IN METRIC SPACES

BY W. A. WILSON

1. *Introduction.* In Menger's studies in metrical geometry* considerable attention is given to the rectification of the simple arc and various definitions of the length of such an arc are discussed. With the definition of arc-length it is then possible to give conditions for the "Konvexifizierbarkeit" of a compact metric space (p. 96) and for the existence of a geodetic arc in a compact metric space (p. 492). Both theorems involve the assumption of the existence of a rectifiable arc between each pair of points. It is intended in this paper to show that these results and some others are due to space properties which are of a more general nature, at least formally, and which suggest possible further studies.

2. *Intrinsic Distance.* If a and b are two points of a metric space Z, we let ab denote the distance between them. A finite set of points $\{a_i\}$ such that $a_0=a$, $a_n=b$, and every $a_ia_{i+1}<\delta$ will be called a δ-chain from a to b, and $aa_1+a_1a_2+\cdots+a_{n-1}b$ will be called its length. If we set $l_\delta(a, b)$ equal to the lower bound of the lengths of all δ-chains from a to b, it is clear that this number exists if there is any such chain, that it is greater than or equal to ab, and that it increases monotonely as $\delta\rightarrow 0$. The upper bound of $l_\delta(a, b)$ for all values of δ is called the *intrinsic distance*† from a to b and is denoted by $l(a, b)$.

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* *Untersuchungen über allgemeine Metrik,* Mathematische Annalen, vol. 100, pp. 75–163 and vol. 103, pp. 466–501. See also Annals of Mathematics, vol. 32, pp. 739–746.

† This turns out to be essentially the same thing as Menger's "*geodetic distance,*" loc. cit., p. 492. See §§4 and 7 below.