A GENERALIZATION OF WEIERSTRASS' AND FEKETE'S MEAN-VALUE THEOREMS*

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1. *Introduction.* Three of our recent papers have dealt with the problem of determining zero-free regions for certain sums of rational functions.† Our study was based largely upon the familiar principle that a vector sum $\sum \overrightarrow{OP_j}$ cannot vanish if all the points P_j lie in the same angle with vertex at O and with a magnitude of less than π radians. In the present paper, we propose to employ the same principle in order to generalize two mean-value theorems.

The first of these theorems will be that of Weierstrass.‡ *If $g(z)$ is real and positive on a curve C: $z=\psi(t)$, $(a \leqq t \leqq b)$, and if $w=f(z)$ maps C one-to-one continuously on a regular curve Γ of the w-plane, any convex region containing Γ also contains the point σ as defined by the equation*

$$\int_a^b f(t)g(t)dt = \sigma \int_a^b g(t)dt.$$

In §2, the above hypothesis on $g(z)$ will be replaced by the more extensive one that $g(z)$ assume any value in a given angular domain with an opening of less than π radians. The point σ will then be free to lie in a region which includes at least the convex region of Weierstrass' theorem. This larger region will be determined in §2 and shown to be a "best approximation" to the position of σ.

The other theorem to be considered is one due to Fekete.§ *If a polynomial $P(z)$ of degree n takes on at $z=k_1$ and $z=k_2$ the unequal values r_1 and r_2, then it takes on every value of the line-*

* Presented to the Society, December 30, 1931.

† This Bulletin, vol. 35 (1929), pp. 363–370, and Transactions of this Society, vol. 32 (1930), pp. 658–668, and vol. 33 (1931), pp. 934–944.

‡ See Osgood, *Lehrbuch der Funktionentheorie*, 1923, vol. I, p. 212.

§ M. Fekete, Acta Universitatis Hungaricae, vol. 1 (1923), pp. 98–100. Also Pólya-Szegö, *Aufgaben und Lehrsätze*, vol. I, p. 257.

segment r_1r_2 *at least once within or on a circle with center at* $z = (k_1+k_2)/2$ *and with a radius* $\frac{1}{2}|k_1-k_2|$ *ctn* $(\pi/(2n))$.

This theorem has been extended by J. v. Sz. Nagy* as follows: *If* $P(z)$ *is a polynomial of degree* n, *if* $P(k_j)=r_j$, $(j=1, 2, \cdots, m)$, *and if* K *and* R *are the smallest convex polygons enclosing the points* k_j *and* r_j, *respectively, then* $P(z)$ *takes on any value* σ *in* R *at least once within the smallest circle from whose circumference* K *subtends an angle* $\phi \leqq \pi/n$.

An equivalent way of expressing Nagy's theorem is that if $P(z)$ is a polynomial of degree n, if $k_1, k_2, \cdots, k_m$ are points of a convex region K, and if $\alpha_1, \alpha_2, \cdots, \alpha_m$ are positive real numbers, then $P(z)$ assumes the value σ, where

$$\sigma \sum_{j=1}^{m} \alpha_j = \sum_{j=1}^{m} \alpha_j r_j,$$

at least once in the smallest circle from whose circumference K subtends an angle $\phi \leqq \pi/n$.

In §3, the above hypothesis on the α_j will be replaced by the less restrictive one that the α_j all lie in an angular domain with an opening not exceeding π. The region in which $P(z)$ will at least once assume the value σ will necessarily be larger than that required under Fekete's or Nagy's assumptions. This larger region will be described in §3. Finally, §4 will be devoted to further discussion of Fekete's work.

2. *Weierstrass' Theorem.* We may state our generalization of Weierstrass' theorem as follows.

THEOREM 1. *Given* $C: z=\psi(t)$, $(a \leqq t \leqq b)$, *a rectifiable curve in the z-plane,* F *a convex region in the w-plane and* G *a region in the w-plane composed of the points lying in or on an angle with vertex at the origin and with a magnitude of* $\gamma < \pi$. *Let* Σ *be the star-shaped region*† *consisting of all points* w *at which* F *subtends an angle of not less than* $\pi - \gamma$.

If $f(z)$ *and* $g(z)$ *are any two functions which on* C *are continuous except for a finite number of finite jumps and which on* C *assume only values within* F *and* G *respectively, then the point* σ *as defined by the equation*

* Jahresbericht der Vereinigung, vol. 32 (1923), pp. 307–309.

† See M. Marden, Transactions of this Society, vol. 32 (1930), pp. 658–9.

$$\int_a^b f(t)g(t)dt = \sigma \int_a^b g(t)dt \tag{1}$$

lies in Σ. *Conversely, if* σ *is any point of* Σ, *functions* $f(z)$ *and* $g(z)$ *fulfilling the above conditions can be found so that equation* (1) *is satisfied.*

In demonstrating this theorem, we may suppose that G is defined by the inequality

$$0 \leqq \arg w \leqq \gamma.$$

No loss of generality ensues since multiplication of equation (1) by $e^{i\theta}$ does not affect the value of σ.

To prove the first part of the theorem, let us assume that σ lies outside of Σ. This means that for all points z on C, the vector $f(z)-\sigma$ lies within an angle of magnitude less than $\pi-\gamma$. That is to say, there exists a positive real number δ such that for all z on C

$$0 \leqq \arg [f(z) - \sigma] - \delta < \pi - \gamma;$$
$$0 \leqq \arg g(z)[f(z) - \sigma] - \delta < \pi.$$

Hence

$$\int_a^b g(t)[f(t) - \sigma]dt \neq 0,$$

in contradiction to the fact that σ satisfies equation (1). Consequently, the point σ must lie in Σ.

To prove the second part of the theorem, let us suppose σ to be any point of Σ. Then σ does or does not also lie in F. If σ lies in F, we need only choose $f(z)=\sigma$ and $g(z)=1$. For then we shall have

$$\int_a^b g(t)[f(t) - \sigma]dt = (b - a)(\sigma - \sigma) = 0.$$

If σ does not lie in F, the angle subtended at σ by F will be $\pi-\gamma'$, where $0\leqq\gamma'\leqq\gamma$. That is to say, there exist in F two points α and β, such that

$$\arg \frac{\beta - \sigma}{\alpha - \sigma} - \pi - \gamma'.$$

Now we have but to make

$$g(z) = \frac{|\beta - \sigma|}{c - a}, \quad f(z) = \alpha, \qquad (a \leqq t \leqq c),$$

$$g(z) = \frac{|\alpha - \sigma|}{b - c} e^{i\gamma'}, \quad f(z) = \beta, \qquad (c \leqq t \leqq b),$$

for then

$$\int_a^b g(t)[f(t) - \sigma]dt = |\beta - \sigma|(\alpha - \sigma) + |\alpha - \sigma|(\beta - \sigma)e^{i\gamma'}.$$

The right-hand side of this expression is zero since

$$\frac{|\beta - \sigma|}{|\alpha - \sigma|} e^{-i\gamma'} = -\frac{\beta - \sigma}{\alpha - \sigma}.$$

In Theorem 1, thus proved, it is to be noted that, when γ approaches zero, region Σ approaches F; that is, our theorem reduces essentially to Weierstrass'; and that, when γ approaches π, region Σ expands indefinitely; that is, the magnitude of the angular domain G cannot be enlarged beyond π without vitally changing the theorem.

Two corollaries may be deduced at once from Theorem 1.

COROLLARY 1. *If $\zeta(z)$ and $F(z)$ are continuous (except for a finite number of finite jumps) on a curve $C: z = \psi(t)$, $(a \leqq t \leqq b)$, and if on C*

$$0 \leqq \arg \zeta(z) \leqq \gamma < \pi,$$

there exists a number σ, $|\sigma| \leqq \sec(\gamma/2)$, such that

$$\int_{t=a}^{t=b} \zeta(z)F(z)dz = \sigma \int_{t=a}^{t=b} \zeta(z)\,|F(z)|\,ds.$$

For let us set in Theorem 1

$$f(z) = e^{i \arg F(z)} \frac{dz}{ds}, \quad g(z) = \zeta(z)\,|F(z)|.$$

Since $|f(z)| = 1$, it follows that $|\sigma| \leqq \sec(\gamma/2)$.

COROLLARY 2. *Under the same conditions as in Corollary 1, if $|F(z)| \leqq M$ on C, a number σ, $|\sigma| \leqq M \sec(\gamma/2)$, exists such that*

$$\int_{t=a}^{t=b} \zeta(z)F(z)dz = \sigma \int_{t=a}^{t=b} \zeta(z)ds.$$

For, let us choose

$$f(z) = F(z)\frac{dz}{ds} \quad \text{and} \quad g(z) = \zeta(z).$$

Then, since $|f(z)| \leqq M$, we have $|\sigma| \leqq M \sec(\gamma/2)$.

These two corollaries lead to the following inequalities:

$$\left|\int_a^b \zeta(z)F(z)dz\right| \leqq \sec\frac{\gamma}{2}\left|\int_a^b \zeta(z)F(z)ds\right|,$$

$$\left|\int_a^b \zeta(z)F(z)dz\right| \leqq M\sec\frac{\gamma}{2}\left|\int_a^b \zeta(z)ds\right|.$$

The last inequality is an extension of the Darboux* theorem that, if $\zeta(z)$ is real and positive on C and ξ a suitable point on C,

$$\left|\int_a^b \zeta(t)F(t)dt\right| \leqq |F(\xi)| \int_a^b \zeta(t)dt.$$

3. *Fekete's Theorem.* Our generalization of Fekete's Theorem and of its extension by Nagy may be put in the following terms.

THEOREM 2A. *Given the positive integer n, the convex region K and the angular domain A with vertex at the origin and with a magnitude of $\gamma<\pi$. Let S be the star-shaped region consisting of all points at which K subtends an angle of not less than $(\pi-\gamma)/n$.*

If $P(z)$ is any polynomial of degree n and if k_j and α_j ($j=1, 2, \cdots, m$) are any points of K and A respectively, then $P(z)$ assumes the value σ, where

$$\sigma\sum_{j=1}^{m}\alpha_j = \sum_{j=1}^{m}\alpha_j P(k_j), \tag{2}$$

at least once in S. Conversely, if s is any point of S, there exists a polynomial $P(z)$ of degree n which for a suitable choice of points k_j and α_j in K and A will satisfy the relation

$$P(s)\sum_{j=1}^{m}\alpha_j = \sum_{j=1}^{m}\alpha_j P(k_j). \tag{3}$$

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* Osgood, *Lehrbuch der Funktionentheorie*, p. 213.

In the proof of this theorem we may suppose without loss of generality that region A is defined by the inequality

$$0 \leqq \arg z \leqq \gamma < \pi.$$

Let us write

$$P(z) - \sigma = C(z - z_1)(z - z_2) \cdots (z - z_n).$$

If all the z_j were outside of S, the rays from any one z_j to all the k_i would lie within an angle of less than $(\pi-\gamma)/n$. That is to say, a constant δ_j would exist such that for all i

$$0 \leqq \arg (k_i - z_j) - \delta_j < \frac{\pi - \gamma}{n},$$

$$0 \leqq \arg [P(k_i) - \sigma] - \arg C - \sum_{j=1}^{m} \delta_j < \pi - \gamma,$$

$$0 \leqq \arg \alpha_i[P(k_i) - \sigma] - \arg C - \sum_{j=1}^{m} \delta_j < \pi,$$

and hence

$$\sum_{i=1}^{m} \alpha_i [P(k_i) - \sigma] \neq 0,$$

in contradiction to the hypothesis that (2) is satisfied. Consequently, $P(z)$ assumes the value σ at least once in S.

To prove the converse proposition, let us choose any point s in S. Point s is or is not also in K. If it is in K, we may choose $m=1$, $\alpha_1=1$, and $k_1=s$, thus satisfying relation (3). If it is not in K, the angle subtended by K at s has a value $(\pi-\gamma')/n$, where $0<\gamma'\leqq\gamma$. That is to say, there exist in K two points k_1 and k_2 such that

$$\arg \frac{k_2 - s}{k_1 - s} = \frac{\pi - \gamma'}{n} \cdot$$

Let us now set $m=2$,

$$\alpha_1 = | k_2 - s |^n, \quad \alpha_2 = | k_1 - s |^n e^{i\gamma'},$$

and

$$P(z) = P(s) + (z - s)^n.$$

Then

$$\alpha_1[P(k_1) - P(s)] + \alpha_2[P(k_2) - P(s)] = |k_2 - s|^n(k_1 - s)^n + |k_1 - s|^n(k_2 - s)^n e^{i\gamma'}.$$

By using the definition of γ', it is obvious that the right-hand side is zero.

It is to be noticed first that, if K is specialized to be a circle whose radius is r, Σ becomes a concentric circle S whose radius is $r \csc [(\pi-\gamma)/(2n)]$.

Secondly, when $\gamma=0$, our theorem provides an approximation to one root of $P(z)-\sigma=0$ which in general is better than Nagy's, coinciding with his only when K is a circle.

Finally, inasmuch as the results in Theorem 2A do not depend upon m, we may state the following theorem.

THEOREM 2B. *Let $P(z)$ be an arbitrary polynomial of degree n and $C: z=\psi(t)$, $(a \leqq t \leqq b)$, a rectifiable curve drawn within a given convex region K. On C let $\alpha(z)$ be continuous and assume only values within an angular domain whose vertex is at the origin and whose magnitude is $\gamma<\pi$. Then the star-shaped region S consisting of all points at which K subtends an angle of not less than $(\pi-\gamma)/n$ contains at least one point s such that*

$$\int_a^b P(t)\alpha(t)dt = P(s)\int_a^b \alpha(t)dt.$$

4. *Addenda*. As Professor Fekete has kindly pointed out to me, he has already proved Theorem 2A (an analog to Bolzano's theorem) for $m=2$.*

Since $\int_a^b\alpha(z)dt \neq 0$ in Theorem 2B, it follows that, if $\int_a^b P(z)\alpha(z)dt=0$, *$P(z)$ vanishes at least once in S*. For $\gamma=0$, this result coincides with one due to Fekete.* For $\alpha(z) \equiv 1$, $\psi(t) \equiv t$ and $P(z)=Q'(z)$ where $Q(z)$ is any polynomial of degree n, it yields an analog to Rolle's theorem also due to Fekete.†

Through a chain of arguments similar to those employed in proving Theorems 2A and 2B, we may establish the following more general theorem.

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* Mathematische Zeitschrift, vol. 22 (1925), p. 2, and Jahresbericht der Vereinigung, vol. 34 (1926), p. 221.

† Mathematische Zeitschrift, vol. 22 (1925), p. 4.

THEOREM 3. *Given two positive integers p and q, a convex region K and an angular domain A with vertex at the origin and with a magnitude of $\gamma<\pi$. Let S be the star-shaped region consisting of all the points from which K subtends an angle of not less than $(\pi-\gamma)/(m+q)$, where $m=\max(p, q)$. Let C: $z=\psi(t)$, $(a\leqq t\leqq b)$, be a rectifiable curve drawn in K, and let $\alpha(z)$ be a function which is continuous on C and which assumes on C only values in A. Finally, let $P(z)$ and $Q(z)$ be any two polynomials of degrees p and q respectively such that $R(z)=P(z)/Q(z)$ is irreducible and has no poles in S. Then in S there exists at least one point s such that*

$$\int_a R(z)\alpha(z)dt=R(s)\int_a^b\alpha(z)dt.$$

Theorem 3 reduces essentially to one due to Fekete* for

$$\psi(t)\equiv k_1,\quad a\leqq t\leqq\tfrac{1}{2}(a+b);\quad \psi(t)\equiv k_2,\quad \tfrac{1}{2}(a+b)<t\leqq b.$$

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* The above Acta article, p. 236.