KRYLOFF ON MATHEMATICAL PHYSICS

Les Méthodes de Solution Approchée des Problèmes de la Physique Mathématique. (Mémorial des Sciences Mathématiques, Fascicule XLIX.) By Nicolas Kryloff. Paris, Gauthier-Villars, 1931. 69 pp.

During the present century two important methods have been developed for dealing with the boundary value problems of mathematical physics: the method of integral equations and the approximation method of W. Ritz. Integral equations are valuable for showing the existence of a solution and determining its analytical properties, but they are not well adapted to the solution of concrete problems where numerical results are wanted. The Ritz method furnishes a direct means of finding a numerical solution that is known or assumed to exist, and is especially adapted to the solution of concrete problems.

In applying the Ritz method to a physical problem, the problem is first set up as a definite integral such that the desired solution makes this integral a minimum (or maximum), just as in the calculus of variations. In the case of a function of a single variable, $y(x)$, the integrand will usually contain y and y'. The Ritz method replaces these by y_m and y_m', where y_m is a linear combination of simple functions, of the form

$$y_m = \sum_{i=1}^{m} a_i\phi_i(x), \quad (m = 1, 2, \cdots, n, \cdots, \infty).$$

Here the $\phi_i(x)$ are easily calculated functions which satisfy the boundary conditions of the problem, as $\sin m\pi x$, $x^n(1-x^2)$, etc. The coefficients a_i are determined from the condition that the integral is to be a minimum. This condition thus gives m equations which must be solved simultaneously for the a's.

Due to the labor of solving a large number of simultaneous equations, it is practically necessary to use only a few terms (not more than 6) of the series assumed for y_m. This circumstance makes it desirable that some means exist for estimating the error committed in stopping at a given number of terms. Although Ritz applied his method to a variety of problems and even solved some that had previously defied solution by all other methods, he left no means of estimating the accuracy of his results.

Since the untimely death of its brilliant author in 1909, the Ritz method has been greatly extended in usefulness and applicability by the researches of Dr. Nicolas Kryloff, and more recently with the collaboration of his pupil and assistant, Dr. N. Bogoliouboff. Kryloff has worked out explicit expressions for the upper limit of the error in the approximate solution (by the Ritz method) of several types of differential equations occurring in mathematical physics, both ordinary and partial.

The book under review gives in rather condensed form the results of Kryloff's researches on the approximate solution of ordinary differential and difference equations of the types most frequently arising in mechanics and physics. The object of the book is "to contribute to the development . . . of methods that make it possible to judge what approximation is to be taken in order that the error committed in that approximation shall be less than a preassigned quantity." The book consists of an introduction of six pages, a first chapter of 41 pages, a second chapter of 15 pages, and an extensive bibliography.

The first chapter deals with the application of the Ritz method to the approximate solution of second-order equations containing a parameter and equations similar to the Euler equation arising in the calculus of variations. The equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = f(x),$$

with

$$y(0) = y(1) = 0, \quad p(x) > 0, \quad q(x) \leqq 0,$$

is first treated. It is shown that the Ritz minimizing suite y_m converges uniformly to the true solution and that the first and second derivatives of y_m converge to the corresponding derivatives of y. The convergence is proved by the method employed by Ritz himself and also by a new method which gives the limit of error at the mth approximation. This latter method gives the error $|y-y_m|$ in terms of $|y-Y_m|$, where Y_m is the sum of the first m terms of the Fourier expansion of y. Explicit expressions are given for the upper limit of $|y-Y_m|$. Indicentally a formula is given for the difference between the Ritz coefficients and the corresponding Fourier coefficients.

The next equation treated is the homogeneous equation containing a parameter,

$$\frac{d}{dx}\left[p\frac{dy}{dx}\right] + \lambda qy + ry = 0, \quad q > 0,$$

with $y(0)=y(1)=0$. In order to estimate the error in the approximate solutions of this equation the author first derives expressions for $|\lambda_k^{(m)}-\lambda_k|$ and $|(\lambda_k^{(m)}-\lambda_k)/\lambda_k|$, where $\lambda_k^{(m)}$ denotes any one of the m roots of the secular equation obtained by equating to zero the determinant of the coefficients of the a's, and λ_k is the corresponding exact value found from a simple expression. These expressions for the error in the parameter enable one to determine in advance just how many terms of the minimizing function y_m to use in order to obtain a given degree of accuracy in the solution. Formulas are next given for computing the possible errors in the functions $y_k^{(m)}$ corresponding to the different values of λ.

The non-homogeneous equation

$$y'' + \lambda qy = f,$$

with $y(0) = y(1) = 0$, is treated in similar fashion and formulas are derived for the possible error in the Ritz solution. The chapter ends with the approximate solution of certain non-linear differential equations.

The second chapter, bearing the title "The Method of Finite Differences," is concerned with the solution of certain difference equations analogous to the differential equations previously treated, and with the same boundary conditions. The methods employed are somewhat similar to those used in the first chapter, except that interpolation formulas are used instead of Ritz functions.

The chapter closes with an article on the approximate solution of differential equations by a method of dividing the interval of integration into m equal parts and considering the coefficients constant in each subinterval. Here, as in all other cases, formulas are given for the possible error in the solution.

Throughout the book the important formulas are numbered with Roman numerals and prominently displayed. The author has supplied a correction to formula (X) as given at the bottom of page 41. The fraction $(m+1)\pi/(\lambda\nu M)$ should be inverted.

The reviewer thinks the usefulness of the book would have been increased if the author had worked out one or two simple examples and shown the reader how to apply the more important formulas to concrete problems. Nevertheless, the book constitutes a valuable addition to the literature of analysis and mathematical physics.

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McCONNELL ON ABSOLUTE CALCULUS

Application of the Absolute Differential Calculus. By A. J. McConnell. London and Glasgow, Blackie and Son, Ltd., 1931. xii+318 pp.

Criticism of this book had best be limited, I think, to the general impression it has made upon me and to my opinion regarding its use as a text in American colleges. In this latter connection I have had a somewhat limited experience, one of my students having studied the book as part of a course in independent reading.

The book is divided into four parts, the first part being devoted to the algebraic preliminaries including such items as the summation convention, coordinate transformations, quadratic forms, tensors, the quotient law of tensors, etc. Part II gives a treatment of the straight line, plane, and cone on the basis of rectangular coordinates; this part also contains a study of affine transformations, including strains and infinitesimal deformations. In Part III we have the differential geometry of curves and surfaces. Part IV contains applications of elementary tensor theory to the dynamics of a particle and rigid bodies, electricity and magnetism, mechanics of continuous media, and ends with a brief account of the special theory of relativity.

An objection which was immediately noticeable to me and one which I do not regard by any means as trivial is McConnell's definition of the tensor (see pp. 2, 20 and 32). McConnell has, in fact, defined the components of a tensor as the tensor itself—the tensor being the object obtained by abstraction from its components with respect to the totality of coordinate systems whose coordinates are related by the underlying family of transformations. Most of us who have written on the subject of tensor analysis have been guilty of using the symbol of the component of the tensor to represent the tensor itself; it is frequently convenient to do this and on one occasion I stated in my writings that this would be done to avoid the multiplicity of symbols which would