# THOMAS HAKON GRONWALL—IN MEMORIAM

With Thomas Hakon Gronwall, who died in New York May 9, 1932, the Society has lost one of its most colorful personalities. By training, habit, and preference a mathematician to whom no chapter of mathematics was a closed book, he could hold his own with physical chemists, and had made a living as a civil engineer specializing in structural steel.

Hakon Tomi Grönwall was born January 16, 1877, at Dylta Bruk in the parish of Axberg in central Sweden. In 1893 he entered the University of Uppsala, but transferred to Stockholm the following year. He was attracted to Stockholms Högskola* by the vigorous and flourishing mathematical school which centered around Mittag-Leffler and which contained among its members mathematicians like Bendixson, Fredholm, von Koch, and Phragmén. That he thrived in this stimulating atmosphere is shown by the ten papers which he wrote in 1895–8†; their range is typical for the broadness of his mathematical interests. He received his doctor's degree at Uppsala at the age of 21; his thesis is [7] in the Bibliography at the end of this obituary. He left Stockholm and Sweden, probably in 1899, owing to a student's prank which was treated by the authorities, sadly lacking in sense of humor, in a way which hurt his youthful pride and his strong love of liberty.

Gronwall now started a completely new chapter. He studied engineering in Berlin, received his degree at the Charlottenburg Technische Hochschule in 1902, and began practising. From 1904 onwards he seems to have been located in this country. Very little is known about this period in Gronwall's life. He was apparently a rolling stone, stopping at the various steel works for longer or shorter periods. He was connected with the American Bridge Co., the Pennsylvania Railroad, and other engineering companies.

In 1911 he lived in Chicago and was evidently ready to make another right-about-face, but now back to his old love. Somehow he managed to keep up with the development in mathematics, so that when he made his spectacular come-back in 1912, it was as a full-fledged mathematician. A number of papers accepted for publication by leading periodicals during the spring of 1912 gave him a desirable confidence in his own abilities. Gronwall made his first appearance before the Society at the Chicago meeting on April 5 and 6, 1912, in which he took a prominent part; and he became a member of the Society during the same month. We can date his return to mathematics from these events.

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* High school is not a correct translation of the Swedish word *Högskola.* In the nineties Stockholms Högskola was essentially a research institute in mathematics and the natural sciences though lacking most of the equipment now-a-days considered necessary for such a task. The student body was small but select. No examinations were held nor degrees given until 1909; a student who desired such baubles had to run the gauntlet at Uppsala. Students who could work under such handicaps had to be earnest seekers of truth.

† These papers are signed H. Grönwall, all later papers have the signature T. H. Gronwall.

During the twenty years of his membership he made nearly ninety communications to the Society. His activities are not evenly distributed in time. The volcanic eruption which started in 1911–2 came to an end in 1916; it flared up again in 1919–20, but faded out after a few years. The fourth period of intense activity started in 1925 and lasted with undiminished strength until his final illness and death. These various periods reflect the vicissitudes in his external circumstances.

Gronwall was called to Princeton in 1913 as an instructor and was adjoined to the editorial board of the Annals. He was made an assistant professor in 1914, but withdrew in 1915. He gave excellent graduate courses, inspired research, and was most useful to the Annals as a contributor and as a critic. Though Gronwall left Princeton long ago, he never lost contact with the Princeton group; he remained on the board of the Annals until 1928 and was a faithful contributor until his premature death. The Annals got more than one third of all his papers, and during the last years of his life nearly all his papers in pure mathematics.

At the next turn of the wheel, we find Gronwall as a mathematical expert on the Technical Staff of the Chief of Ordinance, sharing his time between Washington and the Aberdeen Proving Grounds. This occupation, which lasted from 1918 until 1922, has left some traces in his publications [see 57, 61 and *9*, *11*]. After the demobilization, he established himself as a consulting mathematician in New York. He devoted part of his time to the interests of the American Telephone and Telegraph Co. [see 64], and used to complain that their problems required his knowing Watson's *Theory of Bessel's Functions* by heart. In 1925 he started to collaborate with Professor Victor K. La Mer, which led to his joining the Department of Physics at Columbia University as an associate in 1927. This connection seems to have suited him fully. There were no teaching obligations; he had complete control of his own time and an abundance of new intriguing problems to solve in physical chemistry and in atomic physics. During the last years of his life he worked steadily on the wave mechanics of the helium atom, but was apparently stricken down before he could achieve results of physical significance.*

This sketch of Gronwall's life is necessarily somewhat fragmentary. I shall now proceed to a summary account of his scientific production which I group according to subjects. Gronwall published 85 papers. In addition he communicated the results of 24 investigations to the Society, the details of which are still unpublished. These papers are listed in the Bibliography at the end, reference to which will be made by numbers in square brackets, italic numerals for unpublished, ordinary numerals for published papers. Gronwall was much in demand as a reviewer, having published 33 reviews in this Bulletin.

1. *Algebra and number theory.* [4, 5, 6, 72, 77, and *16*.] This field is not so well represented. His early investigations on transcendental numbers do not contain any new results, the other papers seem to be of minor interest.

2. *Analytical number theory.* [17, 19, 20, 21.] Here we are dealing with

* An inventory of his posthumous papers may prove this statement to be unduly pessimistic.

much more important papers, contributions which helped to make Gronwall famous in 1912–13. In [17] he is concerned with the asymptotic properties of the number theoretical function $\sigma_\alpha(n)$, that is, the sum of the $\alpha$th powers of the divisors of $n$. In [19] and [20] he studies the Riemann zeta function, its reciprocal and its logarithmic derivative on the line $\sigma = 1$. In [21] he applies the same methods in studying the order of magnitude of $L(1, \chi)$ as a function of $k$, where $L(s, \chi)$ is an $L$-function of Dirichlet corresponding to complex characters modulo $k$. Some of Gronwall's estimates have been improved upon later, others still represent the best available.

3. *Functions of a real variable.* [12, 13, 14, 22, 23, 24, 25, 26, 27, 28, 29, 40, 60, 73, 79, 83, and *4, 14.*] With one or two exceptions all these investigations deal with Fourier series, Laplace series, and the theory of summability, in general or as applied to these series. Gronwall devoted several papers to Gibbs' phenomenon and its generalizations. Paper [12] contains a complete discussion of the phenomenon for the partial sums; [14] is a study of the behavior of the partial sums in the case of bounded functions whose Fourier coefficients are $O(1/n)$, where he encountered a similar phenomenon; [73] deals with the $(C, k)$-sums and contains the determination of Cramér's constant. Paper [79] deals with another peculiarity of the $(C, k)$-sums of Fourier or Laplace series. Lebesgue's constants, $\rho_n$, form the subject matter of [13] and [25]. Gronwall proved Fejér's guess that the sequence $\{\rho_n\}$ is monotone. His own guess that the sequence might be completely monotone was proved later by Szegö. Paper [60] contains an interesting complement to Carleman's theorem on the Fourier coefficients of continuous functions. The investigations on the series of Laplace and Legendre [22, 24, 26, and 28] belong to Gronwall's very best papers. Here he determined the order of magnitude of the corresponding Lebesgue constants, constructed functions exhibiting the du Bois-Reymond or the Lebesgue singularity, determined the index of Cesàro summability with careful discussion of the conditions at the antipode or at the end points, and studied the degree of convergence. Most results are final and represent a considerable advance beyond the previous results due to Chapman, Fejér, Haar, and Jackson. Gronwall showed that any series summable $(C, k)$ is also summable de la Vallée Poussin [27] (simultaneously proved by C. N. Moore); his further studies of the $(VP)$-means [40 and *4*] ultimately led him to an elegant investigation of a class of definitions of summation defined by conformal mapping [83].

4. *General theory of functions of one complex variable.* [9, 15, 16, 31, 42, 43, 54, 59, 63, and *8, 17.*] The range of these papers is impressive; their strength lies perhaps less in the novelty of the result than in the elegance of the proofs, however. As an example we may mention [31] and [63], which give new and elementary proofs of important theorems on power series with prescribed initial coefficients.

5. *Special analytic functions.* [36, 41, 44, 47, 66, 84, and *6, 15.*] Gronwall translated Jensen's excellent paper on the gamma function [36]; to this he added his own exposition based on definite integrals [47], and several investigations of special questions in the theory of the gamma and related functions of which [66] is perhaps the most interesting.

6. *Conformal mapping.* [30, 37, 38, 51, 52, 55, 56, 58, and *7.*] The main existence questions having been solved around 1912–13, a study of the fine structure of the mapping function could start. Gronwall took a prominent part in this study. Of particular importance is [30] which involves applications of the area theorem, partly contemporary with, partly preceding the corresponding investigations of Bieberbach. The other notes contain a number of interesting estimates; his claim in [*7*] of having solved the coefficient problem was probably based upon an error, however. It is a pity that Gronwall did not publish a comprehensive presentation of his work in this field.*

7. *Functions of several complex variables.* [1, 2, 3, 7, 45, 62, and *1*, *3*, *11.*] Gronwall's liking for this field goes back to 1895. Paper [1] contains among other things a generalization of Laurent's theorem, restated in [7] and [*3*]; [7] also contains an important contribution to the notion of irreducibility. The theory of abelian functions and integrals plays a great rôle in his early work; [2] gives a mode of constructing integrals of the first kind. But Gronwall's most important contribution to this field is [45] in which he takes up Cousin's work on the expressibility of meromorphic functions as the quotient of entire functions and the extensions of the theorems of Weierstrass and Mittag-Leffler for critical examination; he restates and corrects Cousin's main theorems, and proves by an example that the original formulations were at fault.

8. *Differential equations.* [1, 3, 7, 8, 10, 49, 50, 70, 75, 85, and *2*, *20.*] Here [50] and [70] deal with ordinary differential equations, [49, 75, 85, and *20*] with partial ones. The main interest attaches to his early investigations on systems of $m$ linear total differential equations in $m$ dependent and $n$ independent variables whose general solution can be expressed linearly in terms of a fundamental system of $m^2$ functions. The singularities of the single-valued coefficients are supposed to be the manifolds on which certain entire functions vanish. The Fuchs-Frobenius theory can be extended to this class of equations. Gronwall gives a reduction to normal form, studies the solutions in the neighborhood of a point on one of the singular manifolds, and determines conditions for the manifold to be regular-singular. These investigations complete and extend earlier, partly unsatisfactory, research in the case $n=2$ and rational coefficients due to Fuchs, Horn, and others. In [10] he considers the case in which a point lies on several singular manifolds and obtains a sufficient condition that the corresponding substitutions commute, and proves by an example that Horn's discussion of a special case of this question is erroneous. In several papers he treats the case in which the coefficients are $2n$-tuply periodic and the general solution is single-valued. Gronwall planned at least twice to republish these investigations in more accessible form, but unfortunately never did so. As a consequence they seem to have been overlooked by later writers.

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* Perhaps he wrote such a paper; I have at least a vague recollection of H. von Koch telling me in 1921 that Gronwall had submitted such a memoir to the Acta Mathematica in competition for the prize later awarded to Koebe. The merits or demerits of this memoir and its ultimate fate are matters beyond my ken.

9. *Differential geometry*. [34, 35, 80, 85, and *5, 21*.] The differential geometric questions treated by Gronwall are scarcely of fundamental importance, but they led to difficult analytic problems which required all his skill. For this reason, if for no other, these papers demand our respect.

10. *Mathematical physics* (see also sections 11 and 12). [18, 33, 39, 46, 53, 57, 61, 65, 69, 76, 81, and *9, 10, 11*.] The papers on elasticity [46, 53, 69] reflect the interest he held for such questions since his engineering days. The papers on ballistics [57, 61, and *9, 11*] date from the war period, the paper on induction [64] is an A. T. & T. Co. product. His work in potential theory [18, 81, and *10*] is of considerable interest. Two notes on the kinetic theory of gases [33, 39] and two on optics [65, 76] add to the kaleidoscopic effect.

11. *Atomic physics*. [77, 78, 85, and *22, 23*.] Gronwall's important work in this field was interrupted by his premature death. Papers [77, 78] deal with the wave mechanics of the hydrogen atom; [85] is a mathematical complement to a memoir on the helium atom intended for the Physical Review. A paper elaborating the results in the lattice theory of crystals announced in [*22*] is expected to appear later.

12. *Physical chemistry*. [67, 68, 70, 71, 82 and *18, 24*.] For these questions I am happy to be able to quote from a letter by Professor Victor K. La Mer who writes:

"Physical chemists are deeply indebted to Dr. Gronwall for the interest and rare mathematical skill which he brought to bear in solving the problem of the anomaly of strong electrolytes. This problem has been the outstanding stumbling block of physical chemistry since the original formulation of the dissociation theory by Arrhenius in 1888. In 1923 Debye and Hückel developed a theory which completely revolutionized the point of view. Their theory amounts to determining the correction to the equation of state for a system of electrically charged particles in a medium of dielectric constant $D$ by determining the excess electric potential arising from the unequal distribution of the ions. The theory finds mathematical expression in terms of a combination of the Poisson and Boltzmann equations. Debye and Hückel, however, neglected all terms of higher order than the linear term in the expansion of the exponential in their solution of the problem. By 1925 sufficient experimental evidence had accumulated to convince the writer that the theory would have to be abandoned despite its many remarkable successes unless it could be shown that the very marked discrepancies which frequently persisted to extreme dilutions in the most significant data could be attributed to an incomplete mathematical development. Gronwall immediately grasped the physical significance of the problem; the mathematical aspects were sufficiently difficult to attract his intense interest. Paper [71] contains a complete analytical solution of the Poisson-Boltzmann equation, along with extensive tables for evaluating the numerical solution in the third and fifth approximation valid for the case of salt mixtures of symmetrical valence types. Paper [70] contains an existence proof of the solution and in [82] the complicated numerical methods for the unsymmetric case are completed and confirmed. Gronwall's solution has since been tested extensively on experimental work from this laboratory and elsewhere. It reproduces the results with remarkable fidelity and completely removes the manifest absurdity of negative ion sizes inherent in the original theory. Only a mathematician with Gronwall's gift for analysis and most uncommon grasp of the literature of chemistry and physics could have contributed the elegant solution which he gave. His achievement will remain as a worthy monument of the services which mathematics can render chemistry and physics."

"In our several years of close association, Gronwall was never too busy to instruct the writer or his collaborators in elementary and tedious details of mathematical technique. The skill with which he could analyze and solve difficult problems excited the wonder and admiration of his colleagues. The clarity of his mind and writings have been an inspiration to every one associated with him."

13. *Miscellaneous.* [11, 32, 48, 74, and *13.*] Paper [11] is a fundamental memoir on nomography. The others are minor notes on integral equations, non-euclidean geometry, convex solids, and statistics.

After this survey let us try to analyze Gronwall as a scientific writer. The first impression is undoubtedly that of bewildering versatility. A closer examination reveals, however, that there is much more unity in this work than is apparent at first sight. Gronwall was first and last an analyst, but he frequently went to other fields for questions to turn into analytical problems. His command of the classical tools of analysis was superb; he worked in practically all the main fields of analysis and left a mark for himself in several. His knowledge of the literature, even in remote fields, was remarkable. He was a highly critical reader who got more than one interesting paper out of flaws detected in the current literature. It is idle to speculate over what he might have done, had he done less or lived longer. His life shows that his unruly spirit found expression, joy, and satisfaction in scientific thinking and creation. The average degree of interest of his papers is very high, even those of less intrinsic importance are never empty, and his creations all carry the earmark of an artist who prided himself on doing his work thoroughly, accurately, and with elegance. His papers are usually very well written, he never sacrifices clarity for brevity, and he plays fair with the reader. Gronwall liked to get an elegant solution, but he was able to master great formal complications if necessary. His long and frequent contacts with the applications had given him a strong feeling for what constituted a useful solution in such fields; he was himself an experienced and skilled computer.

When I first met Gronwall some eleven years ago, he was already a disillusioned man, modest, quiet, and retiring. Though he must have been occasionally somewhat of a trial to the puritanical brethren, I have never heard of his having any enemies. His friends found him fair-minded, kind-hearted, and unselfish. He was always willing to put his great knowledge and skill at the service of colleagues, friends, and students, and was eagerly interested in their problems. His non-scientific interests were chiefly intellectual, his acquaintance with general literature was both thorough and wide. His death at the height of his powers means a great loss to science and to his many friends.

## Bibliography

Of the subsequent lists, the first contains the published papers, the second one the abstracts from this Bulletin of the unpublished ones. The abbreviations used in the first list are explained at the end of this paper. The notation [19 (9-11-12) 61] means that this paper was presented to the Society September 11, 1912, an abstract having appeared on page 61 of volume 19 of this Bulletin.

### I. Published Papers


1. *Om system af lineära totala differentialekvationer*, ÖKVAF, vol. 52 (1895), pp. 729–757.
2. *Ueber Integrale algebraischer Differentialausdrücke von mehreren Veränderlichen*, ÖKVAF, vol. 53 (1896), pp. 67–72.
3. *Några användnigar af de 2n-periodiska funktionerna på teorin för system af lineära totala differentialekvationer*, ÖKVAF, vol. 53 (1896), pp. 295–314.
4. *Note sur les fonctions et les nombres algébriques*, ÖKVAF, vol. 54 (1897), pp. 199–203.
5. *Deux théorèmes sur les nombres transcendants*, ÖKVAF, vol. 54 (1897), pp. 623–632.
6. *Sur les nombres transcendants.* II. ÖKVAF, vol. 55 (1898), pp. 153–156.
7. *Om system af linjära totala differentialekvationer särskilt sådana med 2n-periodiska koefficienter.* Akademisk afhandling. Uppsala, 1898, 52 pp.
8. *Eine Verallgemeinerung der Lamé'schen Differentialgleichung*, ÖKVAF, vol. 55 (1898), pp. 357–368.
9. *Sur les fonctions qui ne satisfont à aucune équation différentielle algébrique*, ÖKVAF, vol. 55 (1898), pp. 387–395.
10. *Sur les singularités des systèmes d'équations linéaires aux différentielles totales*, BKSVAH, vol. 24 (I:9) (1899), 22 pp.
11. *Sur les équations entre trois variables représentables par des nomogrammes à points alignés*, JdM, (6), vol. 8 (1912), pp. 59–102.
12. *Über die Gibbssche Erscheinung und die trigonometrischen Summen* $\sin x + \frac{1}{2}\sin 2x + \cdots + (1/n)\sin nx$, MA, vol. 72 (1912), pp. 228–243.
13. *Über die Lebesgueschen Konstanten bei den Fourierschen Reihen*, MA, vol. 72 (1912), pp. 244–261.
14. *On a theorem of Fejér's and an analogon to Gibbs' phenomenon*, TAMS, vol. 13 (1912), pp. 445–468. [18 (4-5-12) 445-6.]
15. *Sur un théorème de M. Picard*, CR, vol. 155 (1912), pp. 764–6. [19 (11-30-12) 225–6.]
16. *On analytic functions of constant modulus on a given contour*, AM, (2), vol. 14 (1912), pp. 72–80. [19 (9-11-12) 61.]
17. *Some asymptotic expressions in the theory of numbers*, TAMS, vol. 14 (1913), pp. 113–122. [19 (4-6-12) 447.]
18. *Some special boundary problems in the theory of harmonic functions*, BAMS, (2), vol. 19 (1913), pp. 227–233. [19 (9-11-12) 61.]

19. *Über das Verhalten der Riemannschen Zetafunktion auf der Geraden $\sigma=1$*, AMP, (3), vol. 20 (1913), pp. 231–8. [19 (11–30–12) 224.]

20. *Sur la fonction $\zeta(s)$ de Riemann au voisinage de $\sigma=1$*, RCMP, vol. 35 (1913), pp. 95–102. [19 (11-30-12) 225.]

21. *Sur les séries de Dirichlet correspondant à des caractères complexes*, RCMP, vol. 35 (1913), pp. 145–159. [19 (11-30-12) 225.]

22. *Über die Laplacesche Reihe*, MA, vol. 74 (1913), pp. 213–270. [19 (9-10-12) 61–2.]

23. *On the summability of Fourier's series*, BAMS, (2), vol. 20 (1913), pp. 139–146. [19 (2-22-13) 392.]

24. *On the degree of convergence of Laplace's series*, TAMS, vol. 15 (1914), pp. 1–30. [19 (2-22-13) 392.]

25. *On Lebesgue's constants in the theory of Fourier series*, AM, (2), vol. 15 (1914), pp. 125–8. [19 (2-22-13) 392.]

26. *Sur la série de Laplace*, CR, vol. 158 (1914), pp. 1488–90. [19 (1-2-13) 299.]

27. *Sur quelques méthodes de sommation et leur application à la série de Fourier*, CR, vol. 158 (1914), pp. 1664–5. [19 (3-21-13) 442.]

28. *Über die Summierbarkeit der Reihen von Laplace und Legendre*, MA, vol. 75 (1914), pp. 321–375. [19 (1-2-13) 299.]

29. *On approximation by trigonometric sums*, BAMS, (2), vol. 21 (1914), pp. 9–14. [20 (12-31-13) 299.]

30. *Some remarks on conformal representation*, AM, (2), vol. 16 (1914), pp. 72–76, and p. 138. [21 (10-31-14) 165.]

31. *On the maximum modulus of an analytic function*, AM, (2), vol. 16 (1914), pp. 77–81. [19 (4-26-13) 507.]

32. *An integral equation of the Volterra type*, AM, (2), vol. 16 (1914), pp. 119–122. [21 (1-2-15) 284.]

33. *A functional equation in the kinetic theory of gases*, AM, (2), vol. 17 (1916), pp. 1–4. [21 (2-24-15) 484.]

34. *Sur les surfaces minima formant une famille de Lamé*, CR, vol. 161 (1915), pp. 582–3. [20 (4-25-14) 517.]

35. *Determination of all triply orthogonal systems containing a family of minimal surfaces*, AM, (2), vol. 17 (1916), pp. 76–100. [20 (4-25-14) 517.]

36. *An elementary exposition of the theory of the gamma function*. Annotated translation of a Danish memoir by J. L. W. V. Jensen. AM, (2), vol. 17 (1916), pp. 124–166.

37. *Sur la déformation dans la représentation conforme*, CR, vol. 162 (1916), pp. 249–252. [21 (1-2-15) 285; 22 (2-26-16) 376.]

38. *Sur la déformation dans la représentation conforme sous des conditions restrictives*, CR, vol. 162 (1916), pp. 316–318. [22 (2-26-16) 376.]

39. *Sur une équation fonctionnelle dans la théorie cinétique des gaz*, CR, vol. 162 (1916), pp. 415–418. [22 (2-26-16) 375.]

40. *Über einige Summationsmethoden und ihre Anwendung auf die Fouriersche Reihe*, JfM, vol. 147 (1916), pp. 16–35. [19 (3-21-13) 442.]

41. *Sur les zéros des fonctions* $P(z)$ *et* $Q(z)$ *associées à la fonction gamma*, ASENS, (3), vol. 33 (1916), pp. 381–393. [22 (2-26-16) 375.]

42. *A problem in geometry connected with the analytic continuation of a power series*, AM, (2), vol. 18 (1916), pp. 65–69. [23 (9-5-16) 81.]

43. *On the power series for* $\log(1+z)$, AM, (2), vol. 18 (1916), pp. 70–73. [23 (9-5-16) 80.]

44. *On the convergence of Binet's factorial series for* $\log \Gamma(s)$ *and* $\psi(s)$, AM, (2), vol. 18 (1916), pp. 74–78. [23 (9-5-16) 81.]

45. *On the expressibility of a uniform function of several variables as the quotient of two functions of entire character*, TAMS, vol. 18 (1917), pp. 50–64. [20 (10-25-13) 173–174.]

46. *Elastic stresses in an infinite solid with a spherical cavity*, AM, (2), vol. 19 (1918), pp. 295–296. [22 (2-26-16) 379.]

47. *The gamma function in the integral calculus*, AM, (2), vol. 20 (1918), pp. 35–124.

48. *On the shortest line between two points in non-euclidean geometry*, AM, (2), vol. 20 (1918), pp. 200–201.

49. *On a system of linear partial differential equations of the hyperbolic type*, AM, (2), vol. 20 (1918), pp. 274–278.

50. *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, AM, (2), vol. 20 (1918), pp. 292–296.

51. *A theorem on power series, with an application to conformal mapping*, PNAS, vol. 5 (1919), pp. 22–24. [25 (12-28-18) 256.]

52. *On the twist in conformal mapping*, PNAS, vol. 5 (1919), pp. 248–250.

53. *On the influence of keyways on the stress distribution in cylindrical shafts*, TAMS, vol. 20 (1919), pp. 234–244. [22 (2-26-16) 379.]

54. *Investigation of a class of fundamental inequalities in the theory of analytic functions.* Annotated translation of a Danish paper by J. L. W. V. Jensen. AM, (2), vol. 21 (1919), pp. 1–29.

55. *On the distortion in conformal mapping when the second coefficient in the mapping function has an assigned value*, PNAS, vol. 6 (1920), pp. 300–302. [26 (2-28-20) 340.]

56. *Conformal mapping of a family of real conics on another*, PNAS, vol. 6 (1920), pp. 312–315.

57. *Qualitative properties of the ballistic trajectory*, AM, (2), vol. 22 (1920), pp. 44–65. [25 (4-26-19) 441; 26 (2-28-20) 340.]

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63. *On power series with positive real part in the unit circle*, AM, (2), vol. 23 (1922), pp. 317–332. [28 (9-8-21) 9–10.]

64. *The mutual inductance of two square coils*, TAMS, vol. 27 (1925), pp. 516–536. [29 (4-28-23) 208.]

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67. *On the extension of the Debye-Hückel theory of strong electrolytes to concentrated solutions.* Written with V. K. La Mer. S, vol. 64 (1926), p. 122. [32 (5-1-26) 316.]

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70. *On the existence and properties of the solutions of a certain differential equation of the second order*, AM, (2), vol. 28 (1927), pp. 355–364. [32 (1-2-26) 127; 33 (2-26-27) 258.]

71. *Über den Einfluss der sogenannten höheren Glieder in der Debye-Hückelschen Theorie der Lösungen starker Elektrolyte.* Written with Victor K. La Mer and Karl Sandved. PZ, vol. 29 (1928), pp. 358–393. [32 (10-30-26) 316.]

72. *The number of arithmetical operations involved in the solution of a system of linear equations*, AMM, vol. 36 (1929), pp. 325–327. [35 (3-30-29) 443.]

73. *Zur Gibbsschen Erscheinung*, AM, (2), vol. 31 (1930), pp. 233–240. [31 (2-28-25) 302; 32 (5-1-26) 316.]

74. *On Minkowski's mixed volume of three convex solids*, AM, (2), vol. 31 (1930), pp. 470–2. [27 (6-17-20) 1, title but no abstract.]

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76. *A formula in geometrical optics*, AM, (2), vol. 31 (1930), pp. 475–478. [36 (10-26-29) 51.]

77. *A diophantine equation connected with the hydrogen spectrum*, PR, (2), vol. 36 (1930), pp. 1671–1672. [33 (10-30-26) 12.]

78. *On the wave equation of the hydrogen atom*, AM, (2), vol. 32 (1931), pp. 47–52. [34 (9-6-28) 703.]

79. *On the Cesàro sums of Fourier's and Laplace's series*, AM, (2), vol. 32 (1931), pp. 53–59. [31 (10-25-24) 109–110.]

80. *A functional equation in differential geometry*, AM, (2), vol. 32 (1931), pp. 313–326. [25 (12-28-18) 255.]

81. *Über den Konvergenzbereich der Potenzreihenentwicklung einer harmonichens Funktion von n Veränderlichen*, MZ, vol. 33 (1931), pp. 177–185. [32 (1-2-26) 127.]

82. *The influence of the higher terms of the Debye-Hückel theory in the case of unsymmetric valence type electrolytes.* Written with V. K. La Mer and Lotti J. Greiff. JPC, vol. 35 (1931), pp. 2245–2288, and pp. 3103–3104.

83. *Summation of series and conformal mapping*, AM, (2), vol. 33 (1932), pp. 101–117. [32 (10-31-25) 36.]

84. *An inequality for the Bessel functions of the first kind with imaginary arguments*, AM, (2), vol. 33 (1932), pp. 275–278. [37 (9-8-31) 530.]

85. *A special conformally euclidean space of three dimensions occurring in wave mechanics*, AM, (2), vol. 33 (1932), pp. 279–293. [37 (9-8-31) 530.]

II. Abstracts of Unpublished Papers

1. *On Weierstrass's preparation theorem.* [19 (1-2-13) 298.]

2. *On systems of linear total differential equations.* [20 (12-31-13) 299.]

3. *Extension of Laurent's theorem to several variables.* [20 (12-31-13) 299.]

4. *On the summation method of de la Vallée Poussin.* [21 (1-2-15) 284.]

5. *Equipotential minimal surfaces.* [25 (12-28-18) 256.]

6. *On Kummer's series.* [25 (12-28-18) 256.]

7. *Upper bounds for the coefficients in conformal mapping.* [27 (9-8-20) 65.]

8. *Some inequalities in the theory of functions of a complex variable.* [27 (10-30-20) 103.]

9. *Some empirical formulas in ballistics.* [27 (2-26-21) 349.]

10. *On biharmonic functions.* [28 (10-29-21) 93.]

11. *Qualitative properties of the ballistic trajectory* (second paper). [28 (2-25-22) 239.]

12. *Isothermal surfaces with spherical lines of curvature in one system.* [29 (4-28-23) 218.]

13. *Extension of Tchebychef's statistical theorem.* [29 (4-28-23) 218.]

14. *Extension of a theorem due to Lerch.* [31 (10-25-24) 109.]

15. *The behavior at infinity of the gamma and associated functions.* [31 (5-2-25) 484.]

16. *The algebraic structure of the formulas in plane trigonometry.* Three papers. [31 (5-2-25) 484; 32 (9-11-25) 23; 32 (10-31-25) 36.]

17. *A new form of the remainder in the binomial series with applications.* Two papers. [32 (10-31-25) 36–37; 32 (2-27-26) 195.]

18. *The variation of the dielectric constant in the Debye-Hückel theory.* Written with V. K. La Mer. [33 (10-30-26) 12.]

*19. On the convergence region of a power series in several variables.* [34 (4-6-28) 423.]

*20. On the differential equation of the vibrating membrane.* [35 (2-23-29) 293.]

*21. Straight line geodesics in Einstein's parallelism geometry.* Two papers. [35 (2-23-29) 293; 35 (3-29-29) 437.]

*22. Some series expansions in the lattice theory of crystals.* [36 (12-27-29) 186.]

*23. Rotational symmetry in the characteristic functions of the Schrödinger wave equation.* [36 (2-22-30) 218.]

*24. On the theory of potentiometric titration.* [37 (9-8-31) 530.]

## List of Abbreviations

AM = Annals of Mathematics

AMM = American Mathematical Monthly

AMP = Archiv der Mathematik und Physik

ASENS = Annales Scientifiques de l'Ecole Normale Supérieure

BAMS = Bulletin of the American Mathematical Society

BKSVAH = Bihang till Kungliga Svenska Vetenskaps Akademiens Handlingar

CR = Comptes Rendus

JdM = Journal de Mathématiques

JfM = Journal für Mathematik

JPC = Journal of Physical Chemistry

MA = Mathematische Annalen

MZ = Mathematische Zeitschrift

ÖKVAF = Öfversigt af Kungliga Vetenskaps Akademiens Förhandlingar

PNAS = Proceedings of the National Academy of Sciences

PR = Physical Review

PZ = Physikalische Zeitschrift

RCMP = Rendiconti del Circolo Matematico di Palermo

S = Science

TAMS = Transactions of the American Mathematical Society

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