where $\mathfrak{D}_i$ is* the relative discriminant of K_{i-1} with respect to K_i. The field K is of degree $\epsilon = e^{(1)} \cdot e^{(2)} \cdot \cdot \cdot e^{(t)}$ with respect to F_2 and by the last reference

$$D = d^{\epsilon} N(\mathfrak{D}),$$

where $\mathfrak{D}$ is the relative discriminant of K with respect to F_2. By the Lemma, every $N(\mathfrak{D}_i) > 1$. It follows that $N(\mathfrak{D}) > 1$. But, by a result due to Chevally, h_2 divides h_1 if there is no field K, $F_1 \geqq K > F_1$, such that the relative discriminant of K with respect to F_2 is of norm unity.† The theorem follows.

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SOME APPLICATIONS OF MURPHY'S THEOREM‡

BY H. BATEMAN

It is well known that the linear partial differential equations of mathematical physics possess solutions in the form of definite integrals with limits depending on the variables entering into the partial differential equations. The law connecting the limits of such an integral with the integrand looks at first sight rather mysterious but the whole matter becomes clear when the integral is expressed as a contour integral with the aid of a theorem due to Murphy and, in a slightly different form, to Cauchy.§

If C is a closed contour containing just one root, a, of the equation $F(x)=0$ and just one root, b, of the equation $G(x)=0$, then, if the radii from these roots turn completely round just once and in one direction as a point describes this contour and if the functions $f(z)$, $\int f(z)dz$, $F(z)$, and $G(z)$ are analytic and uni-

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* Bachmann, loc. cit., p. 452.

† Chevally, *Relation entre le nombre de classes d'un sous-corps et celui d'un sur-corps*, Comptes Rendus, vol. 192 (1931), pp. 257–258.

‡ Presented to the Society, December 27, 1932.

§ R. Murphy, Transactions of the Cambridge Philosophical Society, vol. 3 (1830), p. 429, A. L. Cauchy, Journal de l'École Polytechnique, vol. 12 (1823), p. 580. Murphy's integral has been transformed into a contour integral from which Cauchy's relation may be obtained by an integration by parts.

form at all points inside C and on this contour, we have

$$\int_a^b f(x)dx = \frac{1}{2\pi i}\int_C f(z)dz \log\frac{G(z)}{F(z)} \cdot \tag{1}$$

To apply this theorem to Laplace's equation in n variables,

$$\frac{\partial^2 V}{\partial x_1^2}+\frac{\partial^2 V}{\partial x_2^2}+\cdots+\frac{\partial^2 V}{\partial x_n^2}=0, \tag{2}$$

we make use of the fact that if $c_1, c_2, \cdots, c_n$ are constants satisfying the equation $c_1^2+c_2^2+\cdots+c_n^2=0$, the function

$$V=(x_1^2+x_2^2+\cdots+x_n^2)^{1-n/2} \cdot \log\frac{(x_1-c_1)^2+(x_2-c_2)^2+\cdots+(x_n-c_n)^2}{x_1^2+x_2^2+\cdots+x_n^2}$$

is a particular solution. This is easily verified by noticing that if $r^2=x_1^2+x_2^2+\cdots+x_n^2$, then, for sufficiently large values of r, the function V can be expanded in a power series of type

$$\sum A_p(c_1x_1+c_2x_2+\cdots+c_nx_n)^p r^{2-n-2p},$$

and each term of this series is a solution of the equation.

Generalizing this solution we consider the integral

$$V=\frac{1}{2\pi i}\int_C R^{2-n}f(z)dz\left[\log\frac{H^2}{R^2}-\log\frac{K^2}{R^2}\right],$$

where

$$R^2=\sum_{m=1}^n[x_m-\xi_m(z)]^2,\quad H^2=\sum_{m=1}^n[x_m-c_m(s)]^2,$$

$$K^2=\sum_{m=1}^n[x_m-e_m(t)]^2,$$

and $\xi_m(z)$, $c_m(s)$, $e_m(t)$ are analytic functions of their arguments in suitable domains. The quantities s and t are, moreover, given as functions of z by the equations

$$\sum_{m=1}^n[c_m(s)-\xi_m(z)]^2=0,\quad \sum_{m=1}^n[e_m(t)-\xi_m(z)]^2=0.$$

The contour C is chosen so that $f(z)R^{2-n}$ is analytic in the enclosed realm, so that the equation $R=0$ has no root within it, and so that $\log (H^2/K^2)$ is an analytic function of z which is infinite only at two points within C and returns to its initial value when a point goes round C. We may also consider an integral

$$V = \frac{1}{2\pi i}\int_C R^{2-n} f(z)dz \log \frac{U}{W},$$

where

$$U = \sum_{m=1}^{n} l_m(z)[x_m - u_m(z)], \quad W = \sum_{m=1}^{n} p_m(z)[x_m - v_m(z)],$$

$$\sum_{m=1}^{n}[l_m(z)]^2 = 0, \quad \sum_{m=1}^{n}[p_m(z)]^2 = 0.$$

In both cases the integrand of the integral V is a solution of (2) for all values of z not depending on the variables x.

The results obtained by considering these two integrals imply that the integral

$$V = \int_0^a \frac{f(z)dz}{R^{n-2}}$$

is a solution of equation (2) if a is defined either by the equa tions

$$\sum l_m(a)[x_m - u_m(a)] = 0, \quad \sum [l_m(a)]^2 = 0,$$

or by the equations

$$\sum [x_m - c_m(s)]^2 = 0, \quad \sum [c_m(s) - \xi_m(a)]^2 = 0.$$

These results are well known*; they are generally more useful when applied to the equation of wave motion than when applied to Laplace's equation, because in the former case the limits of the integral may be real.

In the case of the equation

$$F\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)V = 0, \tag{3}$$

where $F(x, y, z)$ is a homogeneous polynomial of degree n in x, y, and z, we may form a solution of type

* H. Bateman, this Bulletin, vol. 24 (1918), p. 296; The Tôhoku Mathematical Journal, vol. 13.

$$V = \frac{1}{2\pi i}\int_C dsf[x\xi(s) + y\eta(s) + z\zeta(s), s] \log \frac{x\xi(s) + y\eta(s) + z\zeta(s) - \phi(s)}{x\xi(s) + y\eta(s) + z\zeta(s) - \psi(s)},$$

where

$$F[\xi(s), \eta(s), \zeta(s)] \equiv 0.$$

An evaluation of the integral by means of Murphy's theorem gives the solution

$$V = \int_a^b f[x\xi(s) + y\eta(s) + z\zeta(s), s]ds,$$

where the limits a and b are defined by the equations

$$x\xi(a) + y\eta(a) + z\zeta(a) - \phi(a) = 0,$$
$$x\xi(b) + y\eta(b) + z\zeta(b) - \psi(b) = 0,$$

respectively. This result has already been established in another way with the aid of an extension of Lagrange's expansion.* Murphy's theorem fails when the limits of the integral are roots of the same equation $F(x)=0$. In this case there is a slight modification of the formula in which a double loop integral is used. If the contour C is replaced by a contour shaped like a figure of eight, with a root of $F(x)=0$ in each oval, the integral

$$\frac{1}{2\pi i}\int_\infty f(z)dz \log F(z) = \int_a^b f(x)dx,$$

where a and b are roots of $F(x)=0$ contained in the ovals of 8. This equation may be derived from Cauchy's equation by an integration by parts.

Murphy's formula needs modification also when the function $f(z)$ is not uniform. The type of modification which is needed will be shown by a consideration of the following example. Let

$$I = \int_{-a}^{a} \frac{\phi(x)dx}{(a^2 - x^2)^{1/2}}.$$

* H. Bateman, Transactions of this Society, vol. 28 (1926), p. 346. The special case in which f is independent of its first argument is given by A. L. Dixon, Messenger of Mathematics, vol. 33 (1904), p. 172.

The first step is to form a uniform integrand by making the substitution $x=a\cos\theta$. We may then write

$$I = \int_0^\pi \phi(a\cos\theta)d\theta = \frac{1}{2\pi i}\int_C \phi(a\cos\tau)\log\left[\frac{\cos(\frac{1}{2}\tau)}{\sin(\frac{1}{2}\tau)}\right]d\tau,$$

where C is a simple contour enclosing the points $\tau=0$ and $\tau=\pi$ but no point of type $\tau=n\pi$ where n is different from 0 and 1. Transforming back to the variable z given by $z=a\cos\tau$, we obtain

$$I = \frac{1}{4\pi i}\int_\Gamma \frac{\phi(z)}{(a^2-z^2)^{1/2}}\log\left(\frac{a+z}{a-z}\right)dz,$$

where Γ is a contour which crosses itself at two points A and B so as to form two enclosed loops one of which contains the point $-a$ and the other the point a. By using a similar type of contour with a loop enclosing the branch point, we may write with $n=3$

$$V = \int_a^b \frac{f(z)dz}{R} = \frac{1}{4\pi i}\int_\Gamma \frac{f(z)dz}{R}\log\frac{H^2}{R^2},$$

where b is a value of z which makes $R^2=0$ and a is a value of z which makes $H^2=0$. Since the integral with limits 0 and a has been shown to be a solution of Laplace's equation, it follows that the integral

$$V = \int_0^a \frac{f(z)dz}{R}$$

is a solution of Laplace's equation. Particular cases of this theorem have been known for some time, but a simple proof of the general theorem has been lacking. A long proof by direct differentiation may be obtained by integrating by parts before differentiating whenever the integrand of an integral becomes infinite at b.

Murphy's formula, in the form in which it was given originally, is a simple generalization of a formula given by Parseval,*

$$y = a - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[1 - e^{ix}f(a+e^{-ix})\right]e^{-ix}dx,$$

* M. A. Parseval, Mémoires des Savants Étrangers de l'Institut de France, vol. 1 (1805), p. 567.

for the root of $y=a+f(y)$ which is expressed by means of Lagrange's expansion. This formula was extended by Poisson,* who showed that

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} F'(a+e^{-ix}) \log \left[1 - e^{ix}f(a+e^{-ix})\right]e^{-ix}dx$$

$$= F'(a)f(a) + \frac{1}{2!}\,\frac{d}{da}[F'(a)\{f(a)\}^2] + \cdots = F(y) - F(a).$$

Poisson also mentions that Cauchy had presented to the Academy of Sciences a memoir on the expression by means of definite integrals of the roots of equations of any degree. This was probably the unpublished memoir of 1819 mentioned by E. Lindelöf in his *Calcul des Rêsidus* (p. 21). There is, however, a memoir with precisely the above title, that was read in 1824 and described briefly in Cauchy's collected works.

Murphy's rule, that if x_1 is a root of $\phi(x)=0$, $\int_0^{x_1} f(x)dx$ is equal to the coefficient of $1/x$ in

$$-f(x)\log\left[\frac{\phi(x)}{x}\right],$$

has been discussed by Whipple,† who also expresses Murphy's theorem by a contour integral in practically the form in which it is used here except that his contour is a particular one.

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* S. D. Poisson, Journal de l'École Polytechnique, vol. 12 (1823), p. 497.

† F. J. W. Whipple, Quarterly Journal of Pure and Applied Mathematics, vol. 40 (1909), p. 368.