A NOTE ON BILINEAR FORMS

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1. *Introduction.* In this note I consider the proof of the follow-ing theorem due to M. Riesz.‡

THEOREM. *Let $M^*(\alpha, \gamma)$ denote the maximum for variable x of*

(1)
$$\frac{\left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\gamma} \right)^{\gamma}}{\left(\sum_{m=1}^{M} C_m \left| x_m \right|^{1/\alpha} \right)^{\alpha}},$$

where the numbers A_{mn}, $(1 \leqq m \leqq M, 1 \leqq n \leqq N)$, are fixed and the numbers B_n, $(1 \leqq n \leqq N)$, and C_m, $(1 \leqq m \leqq M)$, are all positive. Then $\log M^* (\alpha, \gamma)$ is a convex function of the variables (α, γ) in the triangle

$$0 \leqq \gamma \leqq \alpha \leqq 1$$

of the (α, γ) plane.

I show here that it is sufficient to prove the theorem in the case where $\gamma = \alpha$, $(0 \leqq \alpha \leqq 1)$. Suppose the theorem to have been established in this case.

2. *Proof of the Theorem.* Let d_n, $(1 \leqq n \leqq N)$, be a set of posi-tive numbers for which

(2)
$$\sum_{n=1}^{N} B_n d_n = 1.$$

Then, if $\gamma \leqq \alpha$ the numerator of (1) is the maximum, for all sets of d_n satisfying (2), of the expression

(3)
$$\left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\alpha} d_n^{1-\gamma/\alpha} \right)^{\alpha}.$$

Now on any segment of a line contained strictly in the triangle $0 \leq \gamma \leq \alpha \leq 1$, $1 - \gamma/\alpha$ is of the form $\lambda + \mu/\alpha$, where λ and μ are constants. Substituting in (3), we see that the maximum of (1) is the maximum, for varying x and d, of the expression

$$(4) \qquad \left\{ \frac{\sum_{n=1}^{N} B_n' \left| \sum_{m=1}^{M} A_{mn}' x_m \right|^{1/\alpha}}{\sum_{m=1}^{M} C_m \left| x_m \right|^{1/\alpha}} \right\}^{\alpha},$$

where A_{mn}', B_n' vanish whenever $d_n = 0$, and otherwise

$$A_{mn}' = A_{mn} d_n^{\mu}, \quad B_n' = B_n d_n^{\lambda}.$$

Now assuming the theorem to have been already established in the case where $\gamma = \alpha$, we see that, for each fixed set of d_n the logarithm of the maximum of the expression (4) is convex, and hence $\log M^*(\alpha, \gamma)$, as the maximum of a set of convex functions, is also convex. This proves the required result.

3. *A Correction.* I take this opportunity of remarking that a previous paper† of mine, which gives a proof of the theorem of this note, contains an error, which has been pointed out to me independently by Professor Pólya, Professor M. Riesz, and Professor Tamarkin. In fact, in §2 of this paper, I assume, quite unjustifiably, that $\log M^*(\alpha, \gamma)$ is differentiable, while all that we can legitimately assert is its continuity. The argument can be rectified by some complicated, but rather uninteresting, lemmas on convex functions, but I shall not attempt to carry this out here.

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† R. E. A. C. Paley, *A proof of a theorem on bilinear forms*, Journal of the London Mathematical Society, vol. 6 (1931), pp. 226–230.