CONVEX DOMAINS AND LINEAR COMBINATIONS
OF CONTINUOUS FUNCTIONS*

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1. *Introduction. Dines' Problem.* Let there be given n real, continuous and linearly independent functions

$$(1) \qquad \phi_1(x), \phi_2(x), \cdots, \phi_n(x), \qquad (a \leqq x \leqq b),$$

which we may interpret as cartesian coordinates of an arc of curve Γ in the n-space S_n. Let $K = K(\Gamma)$ denote the smallest closed convex domain in S_n containing the arc Γ. The following lemma is due to F. Riesz.†

LEMMA 1. *A point $X = (x_1, x_2, \cdots, x_n)$ belongs to K if and only if its coordinates can be represented in the form*

$$(2) \qquad x_k = \int_a^b \phi_k(x) d\psi(x), \qquad (k = 1, 2, \cdots, n),$$

where $\psi(x)$ is monotonic and non-decreasing in $[a, b]$, with $\psi(a) = 0$, $\psi(b) = 1$.

A characterization of the points which are *interior* to K is given by the following lemma.

LEMMA 2. *A point $X = (x)$ is an interior point of K if and only if its coordinates can be represented in the form*

$$(3) \qquad x_k = \int_a^b p(x)\phi_k(x)dx, \qquad (k = 1, 2, \cdots, n),$$

where $p(x)$ is continuous‡ and positive in $[a, b]$ and $\int_a^b p(x)dx = 1$.

This lemma is proved in §4 below. I wish to show now how by

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means of these lemmas some interesting questions about linear combinations of the functions (1) can be readily answered. We start with the following problem stated and solved by L. L. Dines.*

Under what conditions does every real linear combination

$$(4)\ \Phi(x) = a_1\phi_1(x) + a_2\phi_2(x) + \cdots + a_n\phi_n(x), \quad \left(\sum_1^n a_k^2 > 0\right),$$

change sign in the interval $[a, b]$?

That this should be so, it is obviously necessary and sufficient that every hyperplane

$$(5)\quad \pi(x) \equiv a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0, \quad \left(\sum_1^n a_k^2 > 0\right),$$

passing through the origin O of S_n shall cut† the arc Γ, that is, that the origin $O = (0, 0, \cdots, 0)$ be an *interior* point of K. From this remark and Lemma 2, we immediately derive the following answer.

THEOREM 1 (L. L. DINES). *Every linear combination* (4) *will change sign in* $[a, b]$ *if and only if there exists a positive continuous function* $p(x)$ *with*

$$\int_a^b p(x)\phi_k(x)dx = 0, \quad (k = 1, 2, \cdots, n).$$

By similar reasons every $\Phi(x)$ will vanish somewhere in $[a, b]$ if and only if the origin O is a point of K. Hence Lemma 1 gives the following result.

THEOREM 2. *Every linear combination* (4) *will vanish some-where in* $[a, b]$ *if and only if there exists a monotonic function* $\psi(x)$ *not identically constant in* $[a, b]$ *with*

$$\int_a^b \phi_k(x)d\,\psi_k(x) = 0, \quad (k = 1, 2, \cdots, n).$$

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* L. L. Dines, *A theorem on orthogonal functions with an application to integral inequalities*, Transactions of this Society, vol. 30 (1928), pp. 425–438. See also L. L. Dines, Annals of Mathematics, (2), vol. 28 (1926), pp. 393–395, and N. H. McCoy, this Bulletin, vol. 36 (1930), pp. 878–882.

† That is to say, have points of the arc Γ on both of its sides.

2. *Generalization of Dines' Problem.* A recent article of J. Favard* suggests the following generalization of the problem of Dines: *For what constants $c_1, c_2, \cdots, c_n$, $(\Sigma c_k^2 > 0)$, does every linear combination* (4), *whose coefficients satisfy the relation*

$$(6) \qquad a_1c_1 + a_2c_2 + \cdots + a_nc_n = 0,$$

change sign in $[a, b]$?

An answer to this question is given in the following theorem.

THEOREM 3. *Every linear combination* (4), *whose coefficients satisfy the relation* (6), *changes sign in* $[a, b]$ *if and only if the system of equations*

$$(7) \qquad \int_a^b p(x)\phi_k(x)dx = c_k, \qquad (k = 1, 2, \cdots, n),$$

has a continuous solution $p(x) \neq 0$ *in* $[a, b]$.†

The proof of this theorem requires the following additional geometric lemma whose proof we postpone to the end of this note (§5).

LEMMA 3. (a) *If a straight line* Δ *of* S_n *has no point in common with* K, *then we can pass through* Δ *a hyperplane* π *which also has no point in common with* K, *that is,* π *is a bound of* K. (b) *If* Δ *and* K *have points in common none of which is interior to* K (*we may call* Δ *a line of support of* K), *then we can pass through* Δ *a hyperplane of support* π *of* K.

Let us return to the proof of Theorem 3. Let Δ denote the straight line joining the point $C = (c_1, c_2, \cdots, c_n)$ to the origin O. Lemma 3 gives the following geometric criterion: *Every* $\Phi(x)$, *satisfying* (6), *will change sign in* $[a, b]$ *if and only if* Δ *contains an interior point* P *of* K. Indeed, if Δ contains such a point P, then every hyperplane (5) through Δ will cut the arc Γ, i.e., every $\Phi(x)$ satisfying (6) will change sign in $[a, b]$. Conversely, if Δ contains no point interior to K, then Lemma 3 shows that there exists a linear combination $\Phi(x)$, satisfying (6), which

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* J. Favard, *Sur les zéros réels des polynômes*, Bulletin de la Société Mathématique, vol. 59 (1931), pp. 229–255.

† If every $\Phi(x)$ changes sign in $[a, b]$, irrespective of (6), then system (7) has a required solution $p(x)$ for any constants $c_1, c_2, \cdots, c_n$. Indeed, the origin O is then interior to K.

does not change sign in $[a, b]$. Moreover, if Δ contains interior points of K, it will also contain such a point $P \neq O$. The co-ordinates of P expressed in the form (3) and multiplied by the ratio $\overline{OC}/\overline{OP}(\neq 0)$, yield the characteristic representation (7) of the coordinates of C. Similarly, Theorem 2 is generalized by the following theorem.

THEOREM 4. *We assume that not every linear combination* (4) *vanishes somewhere in* $[a, b]$. *Under this assumption every* $\Phi(x)$, *whose coefficients satisfy* (6), *will vanish somewhere in* $[a, b]$ *if and only if the system*

$$(8) \qquad \int_a^b \phi_k(x)d\psi(x) = c_k, \qquad (k = 1, 2, \cdots, n),$$

has a monotonic solution $\psi(x)$ *not identically constant in* $[a, b]$.

Indeed, the first assumption of this theorem means that O is an exterior point of K. Then relation (6) implies that $\Phi(x)$ vanishes in $[a, b]$ if and only if the straight line Δ passing through O and C has at least one point P in common with K (Lemma 3, case (a)). The coordinates of P expressed in the form (2), and multiplied by the ratio $\overline{OC}/\overline{OP}(\neq 0)$, yield the characteristic representation (8) of the coordinates of C. It is easy to see geometrically that Theorem 4 fails to be always true if we drop the assumption of its first sentence.

3. *Infinite Sequences of Functions.* Let there be given an infinite sequence of linearly independent continuous functions

$$(9) \qquad \phi_0(x) \equiv 1, \quad \phi_1(x), \quad \phi_2(x), \quad \cdots, \qquad (a \leqq x \leqq b),$$

and let

$$(10) \qquad c_0 > 0, c_1, c_2, \cdots,$$

be a given sequence of real constants. Again it is suggested by Favard's paper that we ask the following question. *For what sequences* (10) *does, for every* n, *a linear combination*

$$(11) \qquad \Phi(x) = a_0\phi_0(x) + a_1\phi_1(x) + \cdots + a_n\phi_n(x),$$

whose coefficients satisfy the relation

$$(12) \qquad a_0c_0 + a_1c_1 + \cdots + a_nc_n = 0,$$

always vanish, at least once, in $[a, b]$?

We shall prove the following theorem.

THEOREM 5. *For every n, a linear combination* (11), *whose coefficients satisfy* (12), *will always vanish at least once in* $[a, b]$ *if and only if the system*

$$(13) \qquad \int_a^b \phi_k(x)d\psi(x) = c_k, \qquad (k = 0, 1, 2, \cdots)$$

admits a monotonic solution $\psi(x)$.

The condition of this theorem is obviously sufficient. Its necessity is readily derived from Theorem 4 and a well known theorem of Helly.*

Indeed, on account of Theorem 4, the finite system

$$\int_a^b \phi_k(x)d\psi_n(x) = c_k, \quad (k = 0, 1, 2, \cdots, n),$$

admits, for every n, a monotonic solution $\psi_n(x)$, with $\psi_n(a) = 0$. By Helly's theorem there exists a subsequence $\psi_{n_i}(x)$ which converges to a monotonic solution $\psi(x)$ of the system (13).

For the special sequence of functions

$$(14) \qquad \phi_0(x) = 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2, \quad \cdots,$$

Theorem 5 was given by Favard (loc. cit. p. 244). For this particular sequence (14), Favard proved Theorem 5 also when one or both of the limits a and b are infinite. Our method can not cover these cases, since Theorem 4, on which it is based, fails to be true if $a = -\infty$, $b = +\infty$, $n \geq 3$, for example, if $c_0 = 1$, $c_1 = c_2 = \cdots = c_{n-1} = 0$, $c_n = 1$.†

4. *Proof of Lemma* 2. The point $X = (x)$ given by (3) is interior to K. Indeed, assuming for the moment that X is on the boundary of K, there is no loss of generality (suitable change

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* E. Helly, Sitzungsberichte der Wiener Akademie, vol. 121 IIa (1912), p. 286 and pp. 288–9.

† Indeed, the polynomial $\Phi(x) = a_0 + a_1 x + \cdots + a_n x^n$ necessarily has a real root if $a_0 + a_n = 0$, while the corresponding system

$$\int_{-\infty}^{\infty} d\psi = 1, \quad \int_{-\infty}^{\infty} x d\psi = \int_{-\infty}^{\infty} x^2 d\psi = \cdots = \int_{-\infty}^{\infty} x^{n-1} d\psi = 0, \quad \int_{-\infty}^{\infty} x^n d\psi = 1$$

has no monotonic solution $\psi(x)$.

of axes) if we assume that the hyperplane $x_1 = 0$ is a plane of support of K, that is

$$\phi_1(x) \geqq 0 \text{ for } a \leqq x \leqq b \text{ and } \int_a^b p(x)\phi_1(x)dx = 0.$$

These last conditions, however, are in contradiction with $p(x) > 0$ and $\phi_1(x) \neq 0$ in $[a, b]$. The sufficiency of the condition of Lemma 2 is thus established.

Let $a = 0$, $b = 1$, and let K_0 be the totality of points $X = (x)$ representable in the form (3) with $p(x)$ continuous and positive in $[0, 1]$ and $\int_0^1 p(x)dx = 1$. The point set K_0 is obviously convex. We shall prove now that *every point* $P = (p_1, p_2, \cdots, p_n)$ *of* K *is a limit point of the point set* K_0. Indeed, by Lemma 1 we may write

$$(15) \quad p_k = \int_0^1 \phi_k(x)d\psi(x), \ \psi(x) \text{ monotonic, } \psi(0) = 0, \ \psi(1) = 1,$$

for $k = 1, 2, \cdots, n$. Consider now the polynomial of degree $m(>1)$

$$(16) \quad P_m(x) = \frac{m}{m+2}\left\{B_m(\psi, x) + \frac{1}{m} x^m - \frac{1}{m}(1-x)^m + \frac{1}{m}\right\},$$

where

$$(17) \qquad B_m(\psi, x) = \sum_{\mu=0}^m \psi\left(\frac{\mu}{m}\right)\binom{m}{\mu}x^\mu(1-x)^{m-\mu}$$

is the mth Bernstein polynomial of $\psi(x)$.* From

$$P_m'(x) = \frac{m}{m+2}\left\{m \sum_{\mu=0}^{m-1}\left[\psi\left(\frac{\mu+1}{m}\right) - \psi\left(\frac{\mu}{m}\right)\right]\binom{m-1}{\mu}x^\mu(1-x)^{m-\mu-1} + x^{m-1} + (1-x)^{m-1}\right\},$$

we see that

$$(18) \qquad\qquad\qquad P_m'(x) > 0,$$

* S. Bernstein, Communications Charkow Mathematical Society, (2), vol. 13 (1912), pp. 1–2.

for $0 \le x \le 1$. Moreover, (16) and (17) give

$$(19) \qquad\qquad P_m(0) = 0, \quad P_m(1) = 1.$$

Finally, a fundamental convergence property of the polynomials of Bernstein shows that

$$(20) \qquad\qquad \lim_{m \to \infty} P_m(x) = \psi(x)$$

for every value of x where $\psi(x)$ is continuous. The point

$$X^{(m)}: \qquad x_k{}^{(m)} = \int_0^1 \phi_k(x) P_m{}'(x) dx, \quad (k = 1, 2, \cdots, n),$$

on account of (18) and (19), belongs to the point set K_0; and, by (20) and Helly's theorem, we have $X^{(m)} \to P$ as $m \to \infty$.

If Q is a fixed *interior* point of K, then Q is also *interior* to a suitable polyhedron with $n+1$ vertices $P_0, P_1, \cdots, P_n$, all of which belong to K. By our last result we can find $n+1$ points $X_0, X_1, \cdots, X_n$ of K_0 such that $X_i, (i = 0, 1, \cdots, n)$, is arbitrarily close to P_i. For such a sufficiently close approximation the point Q will clearly be a point of the polyhedron $X_0 X_1 \cdots X_n$, hence, a point of K_0, since K_0 is convex. This concludes the proof of Lemma 2.

5. *Proof of Lemma 3.* (a) Let Δ have no common point with K. Let $\overline{AB}$ be the shortest distance from Δ to K, the point A belonging to Δ and B to K. Then Δ is perpendicular to the segment AB so that the plane π, perpendicular to AB at A, will contain the straight line Δ. This plane π has no point in common with K (see Carathéodory, loc. cit. p. 198).

(b) Let Δ be a line of support of K. Let I be a fixed interior point of K and let P be a common point of Δ and K. Join I and P by a line segment and take a point P' on its extension beyond the point P. Draw through P' the line Δ' parallel to Δ. *Then Δ' and K have no common point.* For if they had a common point Q', then the point of intersection Q of Δ and IQ' would be an interior point of K, contradicting the fact that Δ is a line of support of K. According to (a) we can pass through Δ' a bound π' of K. If now P' approaches P by moving on the line $P'PI$, the corresponding plane π' will move parallel to Δ. For a countable number of suitably chosen positions of P', the correspond-

ing plane π' will converge* to a limiting plane π, which necessarily contains Δ, and which, as a limit of bounds π' of K, is a plane of support of K.

6. *Conclusion.* Finally, I wish to point out two facts. 1. The assumptions that the functions (1) and (9), respectively, are linearly independent are not essential for Theorems 2, 4, and for Theorem 5, respectively. 2. In Lemma 2, the continuity of $p(x)$ can be replaced by the stronger condition that $p(x)$ be a polynomial. To show this one has to change in the proof given in §4 the definition of K_0 which now should be the set of points given by (3), where $p(x)$ is a polynomial, positive in $[a, b]$, with $\int p dx = 1$. Our Theorems 1 and 3 can be changed accordingly, $p(x)$ being now a polynomial.†

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* See Carathéodory's proof (loc. cit. p. 198) for the existence of a plane of support of K passing through a given boundary point of K.

† After this note was in type, I noticed several articles that are closely connected with it. In two articles by S. Kakeya, *On some integral equations*, I and II (Tôhoku Mathematical Journal, vol. 4 (1914), pp. 186–190, and Proceedings of the Tôkyô Mathematical-Physical Society, (2), vol. 8 (1915–16), pp. 83–102), the possibility of finding a solution $p(x)$ of the system (7) is discussed thoroughly, the solution $p(x)$ being subject to various more or less complicated auxiliary conditions. The conditions $p(x) > 0$, $\int p dx = 1$, which lead to my Lemma 2 above, were not considered by Kakeya. A note by M. Fujiwara, *On the system of linear inequalities and linear integral inequality* (Proceedings of the Imperial Academy of Japan, vol. 4 (1928), pp. 330–333) seems to contain (p. 332) a proof of Dines' Theorem 1, similar to the proof given in the present note. It does not actually cover the theorem, however, since Fujiwara derives there not the condition $p(x) > 0$ of Theorem 1, but only the weaker (necessary but not sufficient) condition $p(x) \geq 0$. Finally, I may mention a beautiful note by W. Fenchel, *Geschlossene Raumkurven mit vorgeschriebenem Tangentenbild* (Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 39 (1930), pp. 183–185), where Lemma 2 is proved by a simple elementary method which requires no limiting process whatever, except, of course, ordinary integration.