CONCERNING COMPACT CONTINUA IN CERTAIN SPACES OF R. L. MOORE

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While writing his colloquium book, *Foundations of Point Set Theory*,† R. L. Moore noted that a large body of theorems concerning internal properties of compact continua could be established on the basis of a set of axioms (Axioms 1–5, *Foundations*) insufficient to make the space S itself a subset of a plane. He suggested that possibly every compact continuum M in S was homeomorphic with a compact continuum in the plane. In the present paper it is shown that, with possibly one exception, this is the case. If S is itself compact then it is homeomorphic with a subset of a sphere (possibly the sphere itself). But any compact continuum which is a *proper* subset of S is homeomorphic with a compact continuum in the plane.

THEOREM 1. *If S is a space in which Moore's Axioms* 1–5 *hold true and M is a closed and compact subset of S, then M is homeomorphic with a subset of a sphere. If furthermore M is a proper subset of S, then it is homeomorphic with a subset of a plane.*

Let E and J denote, respectively, a simple domain‡ and its boundary. It will be shown that if L is any circle in a plane and T_1 is any topological transformation of J into L, then there exists a topological transformation T_2 of $E \cdot M+J$ into a subset of L plus its interior, such that for each point P of J, $T_2(P) = T_1(P)$. From this result it readily follows that any closed and compact subset of S is homeomorphic with a subset of a sphere. If M is a closed and compact *proper* subset of S, and P is a point of $S-M$, then the closed and compact point set $M+P$ is homeomorphic with a subset of a sphere, and thus M is homeomorphic with a proper subset of a sphere, and hence homeomorphic with a subset of a plane. Henceforth it will be assumed that M is a closed and compact subset of $E+J$, where E is a simple domain and J is its boundary.

† Colloquium Publications of this Society, vol. XIII. Henceforth this book will be referred to as *Foundations*.

‡ A *simple domain* is a domain bounded by a simple closed curve.

The compact point set $M+J$ is completely separable (I, 19).† Then for every positive integer n there exists a countable subcollection of regions of G_n covering $M+J$ (I, 20). Thus there exists a countable collection G of regions such that for every n the set of regions of the collection G which are regions of G_n covers $M+J$. For each ordered pair (H, K) of regions of the set G for which there exists a pair (p, q) of simple domains such that‡ $H \supset [\bar{p}\cdot(M+J)+$the boundary of $p]$, $p \supset \bar{q}$, and $q \supset \overline{K}$, select some definite pair of simple domains with these properties. If R is any point of $M+J$, and H is any region of G containing R, then there exists a simple domain p containing R such that $H \supset [\bar{p}\cdot(M+J)+$the boundary of $p]$ (IV, 11). There exists a simple closed curve F lying in p and separating R from some point of $S-\bar{p}$. Let q denote the component of $S-F$ which contains R. Then $p \supset \bar{q}$ (III, 1). There exists a region K of the set G containing R and such that $q \supset \overline{K}$. Thus there exists a countable infinity of pairs (H, K) as described above, and with every such pair there can be associated a pair (p, q) of simple domains with properties as described above. Therefore there exists an infinite sequence of pairs (p_1, q_1), (p_2, q_2), $\cdots$, such that (1) for every n, p_n and q_n are simple domains and $p_n \supset \bar{q}_n$, and (2) if H and K are regions of the set G and there exist simple domains u and v such that $H \supset [\bar{u}\cdot(M+J)+$the boundary of $u]$, $u \supset \bar{v}$ and $v \supset \overline{K}$, then there exists an n such that $H \supset [\bar{p}_n\cdot(M+J)$ $+$the boundary of $p_n]$ and $q_n \supset \overline{K}$.

Let A, B, C, and D denote distinct points of J in the order $ABCDA$. Let P_1, P_2, $\cdots$ denote an infinite sequence of points lying on $J-(A+B+C+D)$ and such that the set $P_1+P_2+\cdots$ is everywhere dense on J. Two lemmas will be interpolated at this point.

LEMMA 1. *If α is a double ruling§ of the interior of the simple closed curve $ABCDA$ and E_1 and E_2 are simple domains such that $E_1 \supset \overline{E}_2$, then there exists a simple closed curve L and a double*

† Unless otherwise stated, all references are to *Foundations.* (I, 19) denotes Theorem 19 of Chapter I.

‡ The statement "A contains B," or "B is a subset of A," is written $A \supset B$.

§ For a definition of this term see *Foundations*, pp. 404–405. The statement "α is a double ruling of the interior of $ABCDA$" is to be interpreted as implying that the arcs of one of the single rulings of α are parallel to AB and CD, and those of the other single ruling are parallel to AD and BC. Either complementary domain of $ABCDA$ may be regarded as its *interior*.

ruling β *of the interior of* $ABCDA$ *such that* (1) L *separates* $\bar{E}_2$ *and* $S-E_1$, (2) *every arc of* α *is also an arc of* β, *and* (3) L *is a subset of the sum of the arcs of* β.

Let F denote a finite point set containing A, B, C, and D, every point common to two arcs of α, and every end point of an arc of α. Let H denote $ABCDA$ plus all arcs of α. Let J_i denote the boundary of E_i $(i=1, 2)$. There exist mutually exclusive arcs R_1R_2 and S_1S_2 lying in $E_1-\bar{E}_2$ except that R_i and S_i are on J_i $(i=1, 2)$ (IV, 19 and II, 1). Then (III, 16) there exist domains U and V, arcs $R_1X_1S_1$ and $R_1Y_1S_1$ on J_1, and arcs $R_2X_2S_2$ and $R_2Y_2S_2$ on J_2 such that (1) U is bounded by $R_1X_1S_1+S_1S_2+R_2X_2S_2+R_1R_2$, V is bounded by $R_1Y_1S_1+S_1S_2+R_2Y_2S_2+R_1R_2$, and (2) $U+V+$segment R_1R_2+segment $S_1S_2=E_1-\bar{E}_2$.There exists a point $P(Q)$ on the segment R_1R_2 (S_1S_2) which does not belong to F and such that (1) either $P(Q)$ does not belong to H, or (2) $P(Q)$ belongs to an arc segment which is a subset of both H and R_1R_2 (S_1S_2). It can be shown that there exists an arc $PZ_1Q(PZ_2Q)$ which lies except for P and Q in $U(V)$, contains no point of F and at most a finite number of points of H, and such that if T is a point of H on the segment PZ_1Q (PZ_2Q), then PZ_1Q (PZ_2Q) *crosses at the point* T† that arc of α (or AB, BC, CD, DA) which contains T. Since Z_1P-P is in U and Z_2P-P is in $S-\bar{U}$, it follows that the arc Z_1PZ_2 crosses the arc R_1PR_2 at the point P. Then (IV, 32) R_1PR_2 crosses Z_1PZ_2 at P. Likewise S_1QS_2 crosses Z_1QZ_2 at Q. It follows that the simple closed curve PZ_1QZ_2P separates $\bar{E}_2$ and $S-E_1$. Furthermore, if P (Q) belongs to H then Z_1PZ_2 (Z_1QZ_2) crosses at P (Q) that arc of α (or AB, BC, CD, DA) which contains $P(Q)$.

Let L denote the simple closed curve PZ_1QZ_2P. Let J' denote the boundary of any of the subdivisions into which α divides the interior of $ABCDA$. Let A', B', C', and D' denote the points of F on J' so that on J' the order $A'B'C'D'A'$ obtains. Then the simple closed curve L satisfies, with respect to $A'B'C'D'A'$, the requirements‡ of Theorem 8 of Chapter VI of *Foundations*, and

† See *Foundations*, p. 201.

‡ Furthermore, if X is a point common to L and $A'B'C'D'A'$, then there exist arcs UXV and ZXY, being subsets of L and $A'B'C'D'A'$, respectively, which *cross* at the point X. Without this additional hypothesis, Theorem 8 of Chapter VI fails to hold true. Though stated in Chapter VI, the modified theorem holds true under Axioms 1–5.

there exists a double ruling $\gamma_{J'}$ of $A'B'C'D'A'$ such that every point of L within $A'B'C'D'A'$ lies on some arc of $\gamma_{J'}$. Thus there exists a double ruling of every subdivision into which α divides the interior of $ABCDA$. The arcs of these rulings can be extended so that there results a double ruling β of $ABCDA$ such that (1) every arc of α is also an arc of β, and (2) every arc of $\gamma_{J'}$ (for every J') is a subset of an arc of β. Then L is a subset of the sum of the arcs of β. This completes the proof of Lemma 1.

LEMMA 2. *If P is a point of a simple closed curve $ABCDA$ distinct from A, B, C, and D, and α is a double ruling of the interior of $ABCDA$, then there exists a double ruling β of the interior of $ABCDA$ such that* (1) *every arc of α is an arc of β, and* (2) *the point P is an end point of an arc of β.*

Returning to the proof of Theorem 1 we can show, using Lemmas 1 and 2 repeatedly, that there exists an infinite sequence $\alpha_1, \alpha_2, \cdots$ of double rulings of the interior of $ABCDA$ (consider E as the interior of $ABCDA$) such that, for every i, $(i=1, 2, \cdots)$, (1) every arc of α_i is also an arc of α_{i+1}, (2) there exists a simple closed curve L_i separating $\bar{q}_i$ and $S-p_i$ and being a subset of the sum of the arcs of α_i and (3) the point P_i is an end point of some arc of α_i.

There exists a homeomorphic correspondence between AD (AB) and the set of real numbers x $(0\leqq x\leqq 1)$ such that A corresponds to 0 and D (B) corresponds to 1. If P is any point on AD (AB) let x_p (y_p) denote the number corresponding to P. If P is any point of $M+J$ and UV is an arc not containing P and belonging, for some i, to the double ruling α_i, then if UV is parallel to AB (AD) the point P is said to lie *to the right of UV* (*above UV*) if and only if it is true that of the mutually exclusive domains into which UV divides E, the one having on its boundary the point D (the point B) either contains P or has P on its boundary. Now let P denote any point of $M+J$. From the Dedekind-cut proposition (I, 64), it follows that there exists a unique point Q on AD such that P lies to the right of every arc of α_i $(i=1, 2, \cdots)$, which has an end point on the segment AQ, but does not lie to the right of any arc of α_i which has an end point on the segment QD. Define x_P to be the number x_Q. Similarly there exists a unique point R on AB such that P lies above every arc of α_i $(i=1, 2, \cdots)$, which has an end point on the segment AR, but does not lie above any arc of α_i

which has an end point on the segment RB. Define y_P to be the number y_R. Thus for every point P of $M+J$ there has been defined a pair of numbers (x_P, y_P). Let T denote the transformation of $M+J$ which throws the point P into the point with coordinates (x_P, y_P) in a plane. Then $T(J)$ is a square and $T(M)$ is a subset of a square plus its interior. It will now be shown that T is a topological transformation.

If it is shown that T is a one-to-one transformation, then it will follow that if the point P of $M+J$ is a limit point of the subset N of $M+J$, then $T(P)$ is a limit point of $T(N)$. To show that T is a topological transformation it will, then, be sufficient to show that if P is a point of $M+J$ and N is a closed subset of $M+J$ not containing P, then $T(P)$ is not a limit point of $T(N)$ and does not belong to $T(N)$. Let P denote a point of $M+J$ and N any closed subset of $M+J$ not containing P. Suppose W is a domain containing P. There exists a region H of the set G containing P, lying in W, and containing no point of N. There exists a simple domain u containing P such that $H \supset [\bar{u}\cdot(M+J)+\text{the boundary of } u]$. There exists a simple domain v containing P and such that $u \supset \bar{v}$. There exists a region K of the set G containing P and such that $v \supset \overline{K}$. Then for some integer (say i_1) the simple domains p_{i_1} and q_{i_1} have the properties stated above for u and v, respectively. Furthermore the simple closed curve L_i separates $\bar{q}_{i_1}$ and $S-P_{i_1}$ (and therefore K and $S-H$) and is a subset of the sum of the arcs of α_{i_1}. By a repetition of this argument it can be shown that there exists an integer i_2 $(i_2>i_1)$, such that L_{i_2} separates P from L_{i_1}. Thus there exist two mutually exclusive simple closed curves, L_{i_1} and L_{i_2}, each separating P from N, and each being a subset of the sum of the arcs of α_{i_2}. There exists a positive number e such that if U_1V_1 and U_2V_2 are any arcs of α_{i_2} parallel to AB (AD) with U_1 and U_2 on AD (AB), then $|x_{U_1}-x_{U_2}|>e$ $(|y_{U_1}-y_{U_2}|>e)$. Clearly, then, if Q is any point of N, either $|x_P-x_Q|>e$ or $|y_P-y_Q|>e$. Then the distance from $T(P)$ to $T(Q)$ is greater than e. Hence T is a topological transformation of $M+J$ into a subset of a square plus its interior.

Let K denote the square which is the image of J, that is, $K=T(J)$. Let L denote any circle in the plane and suppose T_1 is a given topological transformation of J into L. Let T' denote the transformation of K into L such that, if P is a point of J, so that $T(P)$ is a point of K and $T_1(P)$ is a point of L, then $T'[T(P)]=T_1(P)$. Clearly T' is a topological transformation. Then it can† be extended so that there results a topological transformation T'' of K plus its interior into L plus its interior such that on K, T'' reduces to T'. Now let P be any point of $M+J$. Let $T_2(P)$ denote the point $T''[T(P)]$. Then T_2 is a topological transformation of $M+J$ into a subset of the circle L plus its interior, and for each point P on J, $T_2(P)=T_1(P)$. This completes the proof of Theorem 1.

THEOREM 2. *If M is a compact continuum which is a proper subset of a space S in which Axioms* 1–5 *hold true, then there exists in a plane a compact continuum M^* homeomorphic with M.*

Theorem 2, which is an immediate consequence of Theorem 1, answers explicitly Moore's question mentioned in the introduction.

AXIOM 5′. *If P is a point of a domain D, then there exists a simple domain E containing P and lying in D.*

The set of Axioms 1, 3, 4, 5′ is clearly as strong as the set 1–5, as Axiom 2 is a consequence of Axioms 5′ and 4, and Axiom 5 is a consequence of Axiom 5′. The new set of axioms is, in fact, stronger, as there exist spaces satisfying Axioms 1–5 in which Axiom 5′ does not hold true. (See the Preface, *Foundations*.)

THEOREM 3. *If S is a space in which Axioms* 1, 3, 4, 5′ *hold true and M is a completely separable subset of S, then M is homeomorphic with a subset of a sphere. If furthermore M is a proper subset of S, then it is homeomorphic with a subset of a plane.*

The proof of Theorem 3 follows closely the proof of Theorem 1. In proving Theorem 1 the fact that M was closed and compact, rather than merely completely separable, was used only to show that if P is a point of M lying in a domain D then there exists a simple domain u containing P and such that $D \supset (\bar{u}\cdot M+$the boundary of $u)$. But Axiom 5′ gives directly a stronger result than this.

† Schoenflies, *Beiträge zur Theorie der Punktmengen*, Mathematische Annalen, vol. 62 (1906), pp. 286–328. See also J. R. Kline, *A new proof of a theorem due to Schoenflies*, Proceedings of the National Academy of Sciences, vol. 6 (1920), pp. 529–531.

There exists a metric space S in which Axioms 1, 3, 4, and 5′ hold true, but such that S is not completely separable. Hence it does not follow that a space S in which Axioms 1, 3, 4, and 5′ hold true is a subset of a plane, even if it is assumed that S is metric.

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ANALOGS OF THE STEINER SURFACE AND THEIR DOUBLE CURVES*

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The equations $x_1:x_2:x_3:x_4=x^n:y^n:z^n:w^n$, where x, y, z, w are linear functions of three homogeneous parameters, represent a rational surface of order n^2. For $n=2$ we have the well known Steiner surface. The particular subject of this paper is the double curve of such a surface and its representation on the plane. A few general properties must first be mentioned.

We take in the plane the reference system $x=0$, $y=0$, $z=0$, and $x+y+z\equiv -w=0$. The diagonals of the quadrilateral are $x+y\equiv -z-w=0$, etc. The vertices of the diagonal triangle are $(1:1:-1:-1)$, $(1:-1:1:-1)$, $(1:-1:-1:1)$, the fourth coordinate being w. Corresponding to the diagonals, the surface has 3 multiple right lines of order n, each meeting two opposite edges of the tetrahedron in points which correspond to a pair of opposite vertices of the quadrilateral. If n is even, the multiple lines are concurrent at $(1:1:1:1)$, which is a point of order $3(n-1)$ for the surface, corresponding to the vertices of the diagonal triangle and to certain pairs of imaginary points when $n>2$. If n is odd, the multiple lines are not concurrent, but are coplanar, meeting two by two at 3 points corresponding to the vertices of the diagonal triangle. The intersection of two multiple lines is then a point of order $2n-1$ for the surface. The class of the surface is always $3(n-1)^2$. The only pinch points are the 6 in which the multiple lines meet the edges of the tetrahedron. Each coordinate plane contains a single curve of order n, and is tangent to the surface along that curve, the order of contact being $n-1$. When n is even the section by a plane through a multiple line meets it in one variable real point, and

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