Integrals of the first kind are defined by the condition that they keep a finite value at every finite or infinite point of the surface. Picard then finds several conditions that must be met in order that there may exist integrals of the first kind. Examples are given of surfaces which do meet these conditions.

Integrals of the second kind for a total differential are those whose value along a path, which is reducible to a point by continuous deformation, is zero. Such integrals exist and may be found but a lengthy discussion shows that *in general* a surface has no integrals of the second kind. Any integral of the rational total differential $Rdx+Sdy$ which does not meet the conditions of the first two kinds is called an integral of the third kind.

One can draw on a surface, with ordinary singularities, particular curves $C_1, C_2, \cdots, C_\rho$, such that there exists no integral of the third kind for the total differential having as specific logarithmic curves the totality of curves C or a part of them but such that there does exist an integral of the third kind having for specific logarithmic curves a $(\rho+1)$th arbitrary curve Γ and the totality of curves C or a part of them.

The two final chapters discuss the double integrals of rational functions $\int\int F(x, y)dx\,dy$ and $\int\int R(x, y, z)dx\,dy$, where $f(x, y, z)=0$.

The book ends with three notes which had previously appeared in mathematical journals.

F. A. FORAKER

FOWLER ON STATISTICAL MECHANICS

Statistical Mechanics. By R. H. Fowler. Cambridge University Press, 1929. 570 pp.

The Adams Prize in the University of Cambridge for 1923–1924 was awarded to Mr. R. H. Fowler for an essay dealing with the properties of matter at high temperatures. The essay was subsequently developed into the extensive systematic treatise before us for review. The author, with the occasional collaboration of other scientists generously acknowledged in his preface, has prepared a detailed survey of a very large portion of the existing theoretical and experimental material concerning the behavior of matter in bulk. The mechanical principles on which the treatment is based are those of the classical and the Bohr-Sommerfeld theories; the essential modifications necessitated by the newer quantum theory are discussed in the final chapter. It should not be supposed, however, that this point of view detracts seriously from the fundamental value of the book; for, essentially the same statistical methods are effective in the new quantum theory as in the others and the results obtained upon the introduction of the Bose-Einstein and Fermi-Dirac statistical weightings are often only slightly different from those obtained in the older quantum theory. If the author should set out to revise his treatment of those instances where the principles of the present quantum theory produce essential changes, he would run today the same risk that he incurred in 1926–1929 of seeing the basic physical principles of his work supplanted almost before the last pages of the manuscript reached the printer.

On the mathematical side, which we shall discuss in some detail, Mr. Fowler has introduced some very welcome improvements based on papers written in collaboration with Professor Darwin (see references in Cambridge Philosophical Society, Proceedings, vol. 21 (1922–23), p. 720). In order to discuss the precise nature of the contribution, we must recall some of the various approaches to the theory of matter in bulk. One can attack the problem in terms of thermodynamics, the Gibbs statistical mechanics, the combinatorial statistical mechanics (employed in the present work), or the Maxwell-Boltzmann statistical mechanics. In thermodynamics one deals directly with inductive knowledge about the energy exchanges between large aggregates of matter and radiation without inquiry into the detailed mechanism of the exchange; but in each of the various statistical methods one treats the behavior of large aggregates as the statistical resultant of the behavior of the constituent individuals, assumed to obey strict mathematical laws and to proceed with little or no interaction. It is in their use of these statistical and dynamical assumptions that the three methods differ, the order in which we have named them above being that of increasing analytical detail in these regards. In the statistical approach, there is a fundamental paradox in supposing determinate behavior for the individuals and statistical behavior for the aggregate, as one had to do before the discovery of the wave mechanics. The attempts to dispose of this paradox by the hypothesis of continuity of path and various related "ergodic" and "quasi-ergodic" hypotheses about the nature of dynamical systems have been left in an entirely unsatisfactory position until the recent conclusive analyses noted below. We should mention in passing that Mr. Fowler quite wisely does not go beyond a statement of the difficulty. In the combinatorial statistical mechanics, which concerns us here, the possible states of the aggregate together with their appropriate statistical weights are determined combinatorially from the possible states of the individuals and their postulated statistical weights; the equilibrium, and to some extent the dynamical, properties of the aggregate are then calculated as statistical averages or maximum values (most probable values). The statistical weights and averages for the aggregate depend upon the dynamical invariants of the problem (such as energy, moment of momentum, numbers of individuals of various types, and so on), and are to be characterized for large values of these parameters. The specific improvements introduced by Mr. Fowler are:

(1) the consistent use of expected or mean values in place of most probable values;

(2) the use of Laplace's method of generating functions coupled with the method of steepest descent to obtain asymptotic formulas for these mean values, in place of the more usual and somewhat indiscriminate use of Stirling's formula.

The mathematical reader may be pardoned for a feeling of surprise that these innovations should not have been introduced somewhat earlier, particularly because physicists have consistently used these methods in other fields; but he will find in Mr. Fowler's recognition of their value a source of real gratification. An interesting point in connection with the method of steepest descent in this theory is that it is more than a convenient mathematical device; it involves the calculation of a significant equilibrium property of the aggregate under consideration. In essence, the method of steepest descent consists in calculating a given contour integral by deforming the contour, in accordance with Cauchy's integral theorem, into one on which the integrand is negligibly small save in the vicinity of a single point $z=\theta$. In the cases which arise here, the number θ is real and satisfies the inequality $0<\theta<1$; and all average values for the aggregate depend upon θ. Since θ thus specifies the equilibrium or average state of the aggregate, it is natural to interpret it as a measure of the temperature; in fact, it is found that the quantity $-(1/k)\log\theta$, where k is Boltzmann's constant, has the properties of temperature on the absolute scale. With these tools, the author proceeds to a systematic development of his subject, discussing aggregates of a generality suitable to the physical problems he desires to treat and pointing out in detail the relations between his general results and those obtained in the other theories mentioned above. Since the exposition is extended over the whole book and is interrupted by the study of numerous physical applications, mathematical readers may find the following summary of the theoretical discussion helpful. Chapters 1 and 2, Fundamental principles with illustrative examples, comparison with the Gibbs statistical mechanics. Chapter 3 (last two sections), Calculation of fluctuations or standard deviations. Chapter 5, Extension to aggregates in which dissociation and combination of individuals can occur. Chapter 6, Derivation of the laws of thermodynamics, and study of the inverse problem of calculating statistics of the individuals from the statistics of the aggregate. Chapters 17 and 18, The connections with the Maxwell-Boltzmann theory and the dynamics of aggregates. Chapter 20, Calculation of the higher moments of deviations. Chapter 21, Modifications necessitated by the new quantum theory. The results of the combinatorial statistical mechanics are essentially independent of the particular laws governing the behavior of the individuals, so long as interaction is neglected. In order to take account of interaction, it is necessary either to apply the methods of perturbation theory or to make a more detailed study of the mechanism of interaction along the lines of the Maxwell-Boltzmann theory. On the other hand, the observed divergence between the behavior of a physical system and that of a model aggregate with interactions suppressed may be analyzed to throw light on the nature of the existing interactions between the individuals of the physical system. The mathematical theory of these problems has not yet assumed a satisfactory form and must be supplemented by those semi-empirical methods in which the genius of the theoretical physicist is revealed. The discussion of these important matters is given in Chapters 8, 9, and 17–19.

At this point, we shall digress momentarily from our consideration of the book to call attention to subsequent progress in statistical mechanics. The most significant advance is the definitive analysis of the "quasi-ergodic" hypothesis, which finally places classical statistical mechanics upon a sure foundation. Since the history of the successive contributions to this analysis is an unusually complicated one, we shall give merely a chronologically arranged list of papers, without comment, as follows: B. O. Koopman, Proceedings of the National Academy of Sciences (vol. 17 (1931), pp. 315–318); T. Carleman, Arkiv for Matematik, Astronomi, och Fysik (vol. 22B (1931–1932), No. 7); G. D.

Birkhoff, Proceedings of the National Academy of Sciences (vol. 17(1931), pp. 650–660); J. von Neumann, Proceedings of the National Academy of Sciences (vol. 18 (1932), pp. 70–82); E. Hopf, Proceedings of the National Academy of Sciences (vol. 18 (1933), pp. 93–100 and pp. 204–209); B. O. Koopman and J. von Neumann, Proceedings of the National Academy of Sciences (vol. 18 (1932), pp. 255–263); J. von Neumann, Proceedings of the National Academy of Sciences (vol. 18 (1922), pp. 263–266); G. D. Birkhoff and B. O. Koopman, Proceedings of the National Academy of Sciences (vol. 18 (1932), pp. 279–282); T. Carleman, Acta Mathematica (vol. 59 (1932), pp. 63–87); J. von Neumann, Annals of Mathematics, ((2), vol. 33, pp. 587–642 and pp. 789–791). Another important contribution has been made by Carleman, Acta Mathematica (vol. 60 (1933), pp. 91–146) in a long paper on a fundamental integro-differential equation of the Maxwell-Boltzmann theory. Finally, the principles of statistical mechanics under the new quantum theory, which have been more clearly formulated and more widely developed since the publication of Mr. Fowler's book, are ably expounded by J. von Neumann (*Mathematische Grundlagen der Quantenmechanik*, Springer, 1933).

It remains for us to comment, however inadequately, upon the physical side of Mr. Fowler's discussion. In fairness, we shall refer the reader to the article of a reviewer more competent in these matters (see J. H. Van Vleck, Science, vol. 70(1929), pp. 41–43). It suffices to say that the author presents a rich diversity of theoretical and experimental material on such subjects as perfect and imperfect gases, chemical reactions, solutions of electrolytes, interatomic forces, problems of stellar atmospheres, and the behavior of matter at high temperatures. His practice of making continual comparisons between theory and experiment cannot be too highly commended. It is the reviewer's opinion, supported by some of the physicists whom he has consulted, that a distinct improvement would have resulted from a similarly consistent practice of stating the physical conditions under which the statistical method can be expected to yield averages with probable errors between prescribed limits. In principle, the student of Mr. Fowler's treatise has the material for such calculations before him; but in practice he would find it difficult and tedious to perform them and would even have to commence his labors by sharpening the fundamental theorem from which the asymptotic formulas of the book are derived.

The reader of this encyclopaedic work cannot fail to be impressed by the remarkable range and power of statistical methods in theoretical physics. It can hardly be doubted that analogous methods have an equally important part to play in domains outside the exact sciences. The reviewer ventures to suggest that various problems of biology and economics present a sufficient similarity to the problem of the behavior of matter in bulk to demand parallel statistical theories along synthetic lines.

In closing, we add one more compliment to the long list of those which the Cambridge University Press has received for the distinctive and beautiful volumes presented under its imprint.

M. H. Stone