CERTAIN PROBLEMS OF CLOSEST APPROXIMATION*

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1. *Introduction*. In connection with the theory of systems of polynomials which are orthogonal and normalized with respect to a given weight function an important question is that of the order of magnitude of the nth polynomial of the sequence as n becomes infinite. In the fundamentally important case of Jacobi polynomials† as well as for the systems of polynomials corresponding to much more general weight functions‡ asymptotic formulas show that the polynomials of the normalized sequence remain bounded, at least in the interior of the interval for which they are constructed. This paper is in part devoted to a much less profound but considerably broader study of upper bounds for the order of magnitude of the polynomials under still more general hypotheses as to the character of the weight function. It is believed to be of interest by reason of the simplicity of the methods employed, and their ready applicability to the obtaining of results with regard to the behavior of the polynomials even at points where the weight function vanishes.

Attention is given also to similar problems in the case of trigonometric sums, which are in some respects more readily accessible to treatment than polynomials.

The rest of the paper is concerned with the convergence of the development of a given function in series of the orthogonal polynomials or trigonometric sums, or more directly, as the terms of the series do not enter explicitly into the calculation, with

* Presented before the Society and Section A of the American Association for the Advancement of Science, by invitation, June 21, 1933, on the occasion of the address by Professor L. Fejér.

† See G. Darboux, *Mémoire sur l'approximation des fonctions de trèsgrands nombres, et sur une classe étendue de développements en série*, Journal de Mathématiques, (3), vol. 4 (1878), pp. 5–56.

‡ See G. Szegö, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Mathematische Annalen, vol. 86 (1922), pp. 114–139; S. Bernstein, *Sur les polynomes orthogonaux relatifs à un segment fini*, Journal de Mathématiques, (9), vol. 9 (1930), pp. 127–177, and vol. 10 (1931), pp. 219–286.

the convergence of the corresponding processes of least-square approximation. More generally, an arbitrary positive power of the error is admitted in place of the square in the criterion of closest approximation. The author has pointed out on various occasions the usefulness of Bernstein's theorem in connection with the study of problems of this sort.* Other writers have made effective use of Hölder's inequality in similar situations.† The present treatment is characterized as to method by the use of Bernstein's theorem, or Markoff's theorem, and Hölder's inequality in succession, and as to results by the comparative liberality of the hypotheses with regard to vanishing of the weight function.

2. *Normalized Trigonometric Sums.* Detailed consideration will be given first, for the sake of simplicity, not to polynomials but to trigonometric sums.

Let $T_n(x)$ be an arbitrary trigonometric sum of the nth order,‡ and s an arbitrary positive number, and let

$$H_{ns} = \int_{-\pi}^{\pi} | T_n(x) |^s dx.$$

Let μ_n be the maximum of $|T_n(x)|$, and let x_0 be a value of x for which $|T_n(x_0)| = \mu_n$. By Bernstein's theorem,

$$| T_n'(x) | \leqq n\mu_n$$

everywhere. For $|x-x_0| \leqq 1/(2n)$, by the law of the mean,

$$| T_n(x) - T_n(x_0) | \leqq \mu_n/2,$$

and

$$| T_n(x) | \geqq \mu_n/2.$$

* See, for example, D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI (hereafter referred to as Colloquium), 1930, Chapter III.

† See, for example, J. Shohat, *On the polynomial and trigonometric approximation of measurable bounded functions on a finite interval*, Mathematische Annalen, vol. 102 (1930), pp. 157–175; J. L. Walsh, *On the overconvergence of sequences of polynomials of best approximation*, Transactions of this Society, vol. 32 (1930), pp. 794–816; W. H. McEwen, *Problems of closest approximation connected with the solution of linear differential equations*, Transactions of this Society, vol. 33 (1931), pp. 979–997.

‡ The words "of the nth order" or "of the nth degree" will be understood throughout to mean of the nth order or of the nth degree *at most*.

As the last relation holds throughout an interval of length $1/n$ at least, it is certain that

$$H_{ns} \geqq (1/n)(\mu_n/2)^s,$$

whence

$$\mu_n \leqq 2(nH_{ns})^{1/s}.$$

The conclusion is as follows.

LEMMA 1. *If $T_n(x)$ is a trigonometric sum of the nth order, if*

$$H_{ns} = \int_{-\pi}^{\pi} | T_n(x) |^s dx,$$

and if μ_n is the maximum of $| T_n(x) |$, then

$$\mu_n \leqq 2(nH_{ns})^{1/s}.$$

This statement may be supplemented by an immediate corollary.

COROLLARY. *If sums $T_n(x)$, each of order indicated by its subscript, are defined for an infinite succession of values of n so that H_{ns} is bounded, for a fixed value of s, then $| T_n(x) |$ has an upper bound of the order of $n^{1/s}$.*

Suppose now that $\rho(x)$ is a summable function of period 2π having a positive lower bound:

$$\rho(x) \geqq v > 0$$

for all values of x. Let $T_n(x)$ be a trigonometric sum of the nth order such that

$$\int_{-\pi}^{\pi} \rho(x)[T_n(x)]^2 dx = 1. \tag{1}$$

Inasmuch as $[\rho(x)]^{-1} \leqq 1/v$,

$$H_{n2} = \int_{-\pi}^{\pi} [T_n(x)]^2 dx \leqq 1/v,$$

and application of the Lemma, through its Corollary, with $s=2$, yields the following result.*

* See also D. Jackson, *Orthogonal trigonometric sums*, presently to be published in the Annals of Mathematics. It is to be noted that the proof does not require that the sums $T_n(x)$ form an orthogonal system.

THEOREM 1. *If $\rho(x)$ is a summable function having a positive lower bound, and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that* (1) *is satisfied, then* $|T_n(x)|$ *has an upper bound of the order of* $n^{1/2}$.

Let the hypothesis with regard to a positive lower bound of $\rho(x)$ be replaced by the less restrictive assumption that $\rho(x)$ is nowhere negative and that $[\rho(x)]^{-r}$ is summable over a period, for some positive (not necessarily integral) value of r. Let $T_n(x)$ again be a trigonometric sum of the nth order satisfying (1). Let

$$s = 2r/(r+1), \qquad p = 2/s = 1 + (1/r) > 1.$$

In the integral

$$H_{ns} = \int_{-\pi}^{\pi} | T_n(x) |^s dx,$$

let the integrand be regarded as the product of the factors $[\rho(x)]^{-1/p}$ and $[\rho(x)]^{1/p} | T_n(x) |^s$. Then, by Hölder's inequality,

$$H_{ns} \leqq \left[\int_{-\pi}^{\pi} \left\{ [\rho(x)]^{-1/p} \right\}^{p/(p-1)} dx \right]^{(p-1)/p}$$

$$\cdot \left[\int_{-\pi}^{\pi} \left\{ [\rho(x)]^{1/p} | T_n(x) |^s \right\}^p dx \right]^{1/p}$$

$$= \left[\int_{-\pi}^{\pi} [\rho(x)]^{-r} dx \right]^{1/(r+1)} \left[\int_{-\pi}^{\pi} \rho(x) [T_n(x)]^2 dx \right]^{1/p}.$$

The first integral in brackets on the right, which exists by hypothesis, is independent of n, and the value of the other integral is 1, by the hypothesis on $T_n(x)$. So H_{ns} is bounded, with $s = 2r/(r+1)$, and the Corollary of the Lemma goes over into the following form.*

THEOREM 2. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable, $r > 0$, and if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n so that* (1) *is satisfied, then* $|T_n(x)|$ *has an upper bound of the order of* $n^{(r+1)/(2r)}$.

* See also D. Jackson, Annals of Mathematics, loc. cit., for the particular case $r = 1$.

3. *Normalized Polynomials, Upper Bounds for Entire Interval.* Let $P_n(x)$ be a polynomial of the nth degree, and μ_n the maximum of $|P_n(x)|$ for $-1 \leqq x \leqq 1$. According to Markoff's theorem

$$| P_n'(x) | \leqq n^2\mu_n$$

for $-1 \leqq x \leqq 1$. Let x_0 be a point of the interval $(-1, 1)$ at which $|P_n(x)| = \mu_n$. If x is a point of $(-1, 1)$ distant from x_0 by not more than $1/(2n^2)$,

$$| P_n(x_0) - P_n(x) | \leqq \mu_n/2, \qquad | P_n(x) | \geqq \mu_n/2.$$

At least one of the intervals

$$x_0 - 1/(2n^2) \leqq x \leqq x_0, \qquad x_0 \leqq x \leqq x_0 + 1/(2n^2),$$

is wholly contained in $(-1, 1)$, and $|P_n(x)| \geqq \mu_n/2$ consequently throughout an interval of length at least $1/(2n^2)$. Hence, if $s > 0$ and

$$H_{ns} = \int_{-1}^{1} | P_n(x) |^s dx,$$

it is certain that

$$H_{ns} \geqq \frac{1}{2n^2}\left(\frac{\mu_n}{2}\right)^s, \; \mu_n \leqq 2(2n^2 H_{ns})^{1/s}.$$

More generally, let

$$H_{ns} = \int_a^b | P_n(x) |^s dx$$

for an arbitrary interval (a, b), and let μ_n be the maximum of $|P_n(x)|$ for $a \leqq x \leqq b$. By virtue of the transformation $y = (2x - a - b)/(b - a)$, $P_n(x)$ is a polynomial of the nth degree in y, $Q_n(y)$, having μ_n as the maximum of its absolute value for $-1 \leqq y \leqq 1$, and

$$\int_{-1}^{1} | Q_n(y) |^s dy = \frac{2}{b-a}\int_a^b | P_n(x) |^s dx = \frac{2H_{ns}}{b-a}.$$

Application of the preceding paragraph to $Q_n(y)$ gives an upper bound for μ_n.

LEMMA 2. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H_{ns} = \int_a^b |P_n(x)|^s dx,$$

and if μ_n *is the maximum of* $|P_n(x)|$ *for* $a \leqq x \leqq b$, *then*

$$\mu_n \leqq 2[4n^2 H_{ns}/(b-a)]^{1/s}.$$

If H_{ns} *is bounded for a succession of polynomials* $P_n(x)$, *with a fixed value of* s, $|P_n(x)|$ *has an upper bound of the order of* $n^{2/s}$ *for* $a \leqq x \leqq b$.

An immediate consequence for $s=2$, after the analogy of Theorem 1, a result which is well known on the basis of other lines of reasoning, is as follows.

THEOREM 3. *If* $\rho(x)$ *is a summable function having a positive lower bound for* $a \leqq x \leqq b$, *and if polynomials* $P_n(x)$ *of the nth degree are constructed for successive values of* n *so that*

$$\int_a^b \rho(x)[P_n(x)]^2 dx = 1, \tag{2}$$

then $|P_n(x)|$ *has an upper bound of the order of* n *for* $a \leqq x \leqq b$.

An obvious generalization may be stated as follows.

COROLLARY. *If* $\rho(x)$ *in Theorem* 3, *non-negative throughout* (a, b), *is assumed to have a positive lower bound merely for* $c \leqq x \leqq d$, *where* $a \leqq c < d \leqq b$, *without being restricted as to its vanishing outside* (c, d), *then* $|P_n(x)|$ *has an upper bound of the order of* n *for* $c \leqq x \leqq d$.

For

$$\int_c^d \rho(x)[P_n(x)]^2 dx \leqq \int_a^b \rho(x)[P_n(x)]^2 dx = 1,$$

and the proof of the theorem can be applied directly to the interval (c, d).

The reasoning with Hölder's inequality by means of which Theorem 2 was proved leads now to the following theorem.

THEOREM 4. *If* $\rho(x)$ *is a non-negative summable function such that* $[\rho(x)]^{-r}$ *is summable over* (a, b), *with* $r>0$, *and if polynomials* $P_n(x)$ *of the nth degree are constructed for successive values of* n *so that* (2) *is satisfied, then* $|P_n(x)|$ *has an upper bound of the order of* $n^{(r+1)/r}$ *for* $a \leqq x \leqq b$.

The proof consists in showing, by adaptation of the formulas that we have previously employed, that H_{ns} again is bounded with $s=2r/(r+1)$.

To the result just formulated we may add the following statement.

COROLLARY. *If $\rho(x)$ in Theorem 4 is non-negative and summable over (a, b), and if $[\rho(x)]^{-r}$ is assumed merely to be summable over (c, d), where $a \leqq c < d \leqq b$, then $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/r}$ for $c \leqq x \leqq d$.*

4. *Normalized Polynomials, Upper Bounds for Entire Interval by Trigonometric Substitution.* Let $P_n(x)$ again be a polynomial of the nth degree, and μ_n the maximum of its absolute value for $-1 \leqq x \leqq 1$. Let

$$H'_{ns} = \int_{-1}^{1} (1 - x^2)^{-1/2} | P_n(x) |^s dx,$$

still with the understanding that $s>0$. By the substitution $x=\cos\theta$ the integral becomes

$$H'_{ns} = \int_0^{\pi} | P_n(\cos\theta) |^s d\theta = \frac{1}{2}\int_{-\pi}^{\pi} | P_n(\cos\theta) |^s d\theta.$$

As $P_n(\cos\theta)$ is a trigonometric sum of the nth order in θ, Lemma 1 is applicable, with the conclusion that $\mu_n \leqq 2(2nH'_{ns})^{1/s}$.

For a general interval, let

$$H'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2} | P_n(x) |^s dx.$$

If $y=(2x-a-b)/(b-a)$ and $P_n(x)=Q_n(y)$,

$$\int_{-1}^{1} (1 - y^2)^{-1/2} | Q_n(y) |^s dy = H'_{ns}.$$

Application of the preceding paragraph to this integral gives the following lemma.

LEMMA 3. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2} | P_n(x) |^s dx,$$

and if μ_n is the maximum of $|P_n(x)|$ for $a \leqq x \leqq b$, then

$$\mu_n \leqq 2(2nH'_{ns})^{1/s}.$$

If H_{ns}' is bounded for a succession of polynomials $P_n(x)$, with fixed s, $|P_n(x)|$ has an upper bound of the order of $n^{1/s}$ for $a \leqq x \leqq b$.

For $s=2$ this yields the following result.

THEOREM 5. *If $\rho(x)$ is a summable function such that $\rho(x)[(b-x)\ (x-a)]^{1/2}$ has a positive lower bound over (a, b), and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, then $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ for $a \leqq x \leqq b$.*

COROLLARY. *If $\rho(x)$ is non-negative and summable over (a, b), and if $\rho(x)[(d-x)\ (x-c)]^{1/2}$ has a positive lower bound over (c, d), where $a \leqq c < d \leqq b$, then $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ for $c \leqq x \leqq d$.*

Hölder's inequality is to be applied this time to the integral of the product of the factors

$$[\rho(x)]^{-1/p}[(b-x)(x-a)]^{-1/2}, \quad [\rho(x)]^{1/p} |P_n(x)|^s,$$

where s and p are related to r as before, to obtain the following theorem.

THEOREM 6. *If $\rho(x)$ is a non-negative summable function such that*

$$[\rho(x)]^{-r}[(b-x)(x-a)]^{-(r+1)/2}$$

is summable over (a, b), with $r>0$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, then $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ for $a \leqq x \leqq b$.

It is perhaps not necessary to state the corollary relating to an interval (c, d) contained in (a, b).

5. *Normalized Polynomials, Upper Bounds for Interior of Interval.* The trigonometric substitution of §4 is useful also in connection with the integral denoted by H_{ns} in §3. Let

$$H_{ns} = \int_{-1}^{1} |P_n(x)|^s dx$$

once more, and let $x = \cos\theta$. Then

$$H_{ns} = \int_0^{\pi} \sin\theta\, |P_n(\cos\theta)|^s d\theta.$$

Let N be the smallest integer satisfying the condition that $N \geqq 1/s$. This relation means that $1 \leqq Ns$, so that $\sin\theta \geqq (\sin\theta)^{Ns}$ for $0 \leqq \theta \leqq \pi$, and

$$H_{ns} \geqq \int_0^\pi |\sin^N\theta\, P_n(\cos\theta)|^s d\theta = \frac{1}{2}\int_{-\pi}^{\pi} |\sin^N\theta\, P_n(\cos\theta)|^s d\theta.$$

Let μ_n' be the maximum of $|\sin^N\theta\, P_n(\cos\theta)|$. As the expression in bars is a trigonometric sum of order $n+N$, it follows from Lemma 1 that

$$\mu_n' \leqq 2[2(n+N)H_{ns}]^{1/s}.$$

Hence

$$|P_n(\cos\theta)| \leqq \frac{2[2(n+N)H_{ns}]^{1/s}}{|\sin^N\theta|},$$

or, in terms of the original variable x,

$$|P_n(x)| \leqq \frac{2[2(n+N)H_{ns}]^{1/s}}{(1-x^2)^{N/2}},$$

for $-1<x<1$. With the aid of a further change of variable the conclusion may be expressed in the following form.

LEMMA 4. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$H_{ns} = \int_a^b |P_n(x)|^s dx,$$

and if N is the smallest integer $\geqq 1/s$, then

$$|P_n(x)| \leqq \frac{K[(n+N)H_{ns}]^{1/s}}{[(b-x)(x-a)]^{N/2}}$$

for $a<x<b$, where

$$K = 2\cdot 2^{1/s}[(b-a)/2]^{N-(1/s)}.$$

The significance of this result is that for a fixed x interior to (a, b), or for any closed interval interior to (a, b), the upper bound obtained is of the order of $(nH_{ns})^{1/s}$ for fixed s, as compared with $(n^2 H_{ns})^{1/s}$ in Lemma 2 or $(nH_{ns}')^{1/s}$ in Lemma 3; the earlier lemmas, on the other hand, apply to the entire closed interval $a\leqq x\leqq b$.

It may be noted, though perhaps as a point of minor interest, that the statement of Lemma 4 becomes somewhat simpler and more compact if s is the reciprocal of an integer, so that $N=1/s$.

Application of the Lemma is to be made through the following corollary.

COROLLARY. *If H_{ns} is bounded for a succession of polynomials $P_n(x)$, with a fixed value of s, then $|P_n(x)|$ has an upper bound of the order of $n^{1/s}$ throughout any closed interval interior to (a, b).*

For $s=2$ there is obtained the following well known supplement to Theorem 3.

THEOREM 7. *If $\rho(x)$ is a summable function having a positive lower bound for $a \leqq x \leqq b$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any closed interval interior to (a, b).*

If $\rho(x)$, non-negative throughout (a, b), has a positive lower bound in an interval (c, d) contained in (a, b), $|P_n(x)|$ has an upper bound of the order of $n^{1/2}$ throughout any closed interval interior to (c, d).

A similar continuation of Theorem 4 requires no new calculation; it is a question merely of combining Lemma 4, through its Corollary, with the observation already made as to the boundedness of H_{ns} for $s=2r/(r+1)$.

THEOREM 8. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable over (a, b), with $r>0$, and if polynomials $P_n(x)$ are constructed so that (2) is satisfied, $|P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(2r)}$ throughout any closed interval interior to (a, b).*

This also can be adapted to the hypothesis that $[\rho(x)]^{-r}$ is summable merely over an interval (c, d) contained in (a, b).

6. *Convergence of Trigonometric Approximation.* A schedule of propositions corresponding to those listed above can be worked out with reference to the convergence of trigonometric or polynomial approximations determined by the minimizing of an integral containing a power of the error. In the case of least squares, as is well known, the approximating functions can be regarded as partial sums of developments in series of polynomials or trigonometric sums orthogonal with respect to the weight function in question. The terms of the series will not be explicitly in evidence, however; the treatment of convergence, for least squares as well as in the case of other powers, will be based directly on consideration of the magnitude of the integral which is minimized.

As in the earlier part of the paper, attention will first be given to trigonometric sums.

Let $f(x)$ be a given continuous function of period 2π, and let $T_n(x)$ and $t_n(x)$ be arbitrary trigonometric sums of the nth order. For a given positive s, let

$$G_{ns} = \int_{-\pi}^{\pi} | f(x) - T_n(x) |^s dx.$$

Let the difference $f(x) - t_n(x)$ be denoted by $r_n(x)$, and let ϵ_n be an upper bound for $|r_n(x)|$:

$$| f(x) - t_n(x) | \leqq \epsilon_n.$$

Let $T_n(x) - t_n(x) = \tau_n(x)$, so that $f(x) - T_n(x) \equiv r_n(x) - \tau_n(x)$. Let μ_n be the maximum value of $|\tau_n(x)|$, taken on for $x = x_0$. For $|x - x_0| \leqq 1/(2n)$, inasmuch as $|\tau_n'(x)| \leqq n\mu_n$, by Bernstein's theorem, $|\tau_n(x)|$ remains greater than or equal to $\mu_n/2$. If $\mu_n \geqq 4\epsilon_n$, so that $|r_n(x)| \leqq \mu_n/4$, then

$$| r_n(x) - \tau_n(x) | \geqq \mu_n/4$$

throughout the specified interval of length $1/n$, and

$$G_{ns} = \int_{-\pi}^{\pi} | r_n(x) - \tau_n(x) |^s dx \geqq \frac{1}{n}\left(\frac{\mu_n}{4}\right)^s,$$

from which it follows that

$$\mu_n \leqq 4(nG_{ns})^{1/s}.$$

If the condition $\mu_n \geqq 4\epsilon_n$ is not satisfied, this fact of itself gives an upper bound for μ_n. In any case μ_n has one or the other of the numbers $4\epsilon_n$, $4(nG_{ns})^{1/s}$ for an upper bound, and can not exceed their sum:

$$\mu_n \leqq 4(nG_{ns})^{1/s} + 4\epsilon_n.$$

Since $|r_n(x)| \leqq \epsilon_n$,

$$| f(x) - T_n(x) | = | r_n(x) - \tau_n(x) | \leqq \epsilon_n + \mu_n \leqq 4(nG_{ns})^{1/s} + 5\epsilon_n.$$

The conclusion may be expressed in the following form:*

LEMMA 5. *If $f(x)$ is a continuous function of period 2π, $T_n(x)$ a trigonometric sum of the nth order, and*

* See Colloquium, pp. 84, 87–88.

$$G_{ns} = \int_{-\pi}^{\pi} | f(x) - T_n(x) |^s dx,$$

and if there exists a trigonometric sum $t_n(x)$ of the nth order such that

$$| f(x) - t_n(x) | \leqq \epsilon_n$$

everywhere, then

$$| f(x) - T_n(x) | \leqq 4(nG_{ns})^{1/s} + 5\epsilon_n$$

for all values of x.

Let $\rho(x)$ be a summable function of period 2π having a positive lower bound, $\rho(x) \geqq v > 0$, and let $T_n(x)$ be determined among all trigonometric sums of the nth order as one for which the integral

$$\int_{-\pi}^{\pi} \rho(x) | f(x) - T_n(x) |^m dx \tag{3}$$

has its minimum value, the exponent m being a given positive number. The question of the existence and uniqueness or non-uniqueness of the minimizing sums, for both trigonometric and polynomial approximation, has been treated extensively elsewhere,* and will not be discussed here. Let the minimum value of the integral be denoted by γ_n. In consequence of the hypothesis on $\rho(x)$,

$$G_{nm} = \int_{-\pi}^{\pi} | f(x) - T_n(x) |^m dx \leqq \gamma_n/v,$$

and by application of Lemma 5

$$| f(x) - T_n(x) | \leqq 4(n\gamma_n/v)^{1/m} + 5\epsilon_n,$$

if $f(x)$ can be approximated by a trigonometric sum $t_n(x)$ of the nth order with a maximum error not exceeding ϵ_n.

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* See, for example, D. Jackson, *Note on an ambiguous case of approximation*, Transactions of this Society, vol. 25 (1923), pp. 333–337; *A generalized problem in weighted approximation*, Transactions of this Society, vol. 26 (1924), pp. 133–154; *Note on the convergence of a sequence of approximating polynomials*, this Bulletin, vol. 37 (1931), pp. 69–72; and other passages referred to in these papers.

Under the same assumption as to the existence of an approximating sum $t_n(x)$ an upper bound can be assigned for γ_n. If the integral of $\rho(x)$ over a period is denoted by R, then by virtue of the minimizing property of $T_n(x)$

$$\begin{aligned}\gamma_n &= \int_{-\pi}^{\pi} \rho(x) \mid f(x) - T_n(x) \mid^m dx \\ &\leqq \int_{-\pi}^{\pi} \rho(x) \mid f(x) - t_n(x) \mid^m dx \leqq R\epsilon_n{}^m.\end{aligned}$$

With the incidental observation that $5\epsilon_n \leqq 5n^{1/m}\epsilon_n$, it thus becomes apparent that

$$\mid f(x) - T_n(x) \mid \leqq Cn^{1/m}\epsilon_n,$$

where C is independent of x and independent of n.

The minimizing sums $T_n(x)$ will converge uniformly toward the function $f(x)$ as n becomes infinite if sums $t_n(x)$ exist so that $\lim_{n\to\infty} n^{1/m}\epsilon_n = 0$. A well known sufficient condition,* when $m > 1$, is that $f(x)$ have a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0} \omega(\delta)/\delta^{1/m} = 0$. In this theorem and succeeding theorems, however, the formal statement will be in terms of the order of magnitude of ϵ_n, and reference will be made to systematic presentations of the theory of approximation by polynomials and trigonometric sums† for relations between the values attainable for ϵ_n and properties of continuity of $f(x)$.

THEOREM 9. *If $\rho(x)$ is a summable function having a positive lower bound, if trigonometric sums $T_n(x)$ of the nth order are constructed for successive values of n to minimize the integral* (3), *and if there exist trigonometric sums $t_n(x)$, likewise of the nth order, so that*

$$\mid f(x) - t_n(x) \mid \leqq \epsilon_n,$$

there is a constant C, independent of x and n, such that

$$\mid f(x) - T_n(x) \mid \leqq Cn^{1/m}\epsilon_n.$$

* See Colloquium, p. 89, where a more general form of convergence theorem is given, applying to other sums than those which minimize the error integral. The more general form of statement could be carried through the present paper, but the specific statements with regard to minimizing sums are preferred for the sake of simplicity.

† For example, Colloquium, Chapter I.

The minimizing sums $T_n(x)$ will converge uniformly toward $f(x)$ if sums $t_n(x)$ exist for which $\lim_{n\to\infty} n^{1/m}\,\epsilon_n = 0$.

A new theorem is obtained if it is assumed merely that $\rho(x)$, in addition to being non-negative and summable, is such that $[\rho(x)]^{-r}$ is summable over a period, for a positive value of r. Let

$$s = mr/(r+1), \qquad p = m/s = 1 + (1/r) > 1,$$

and let Hölder's inequality be applied to the integral of the product of the factors

$$[\rho(x)]^{-1/p}, \qquad [\rho(x)]^{1/p}\,|\,f(x) - T_n(x)\,|^s,$$

to give a comparison between this integral and a product of integrals involving these factors with the exponents $p/(p-1)$ and p, respectively. The resulting inequality is

$$\begin{aligned} G_{ns} &= \int_{-\pi}^{\pi} |\,f(x) - T_n(x)\,|^s dx \\ &\leqq \left[\int_{-\pi}^{\pi} [\rho(x)]^{-r} dx\right]^{1/(r+1)} \\ &\qquad \cdot \left[\int_{-\pi}^{\pi} \rho(x)\,|\,f(x) - T_n(x)\,|^m dx\right]^{r/(r+1)} . \end{aligned}$$

If γ_n and ϵ_n are used with the same meanings as before, the last member does not exceed a constant multiple of $\gamma_n^{r/(r+1)}$ or of $\epsilon_n^{mr/(r+1)} = \epsilon_n^s$, and Lemma 5 gives for $|f(x) - T_n(x)|$ an upper bound of the order of $n^{1/s}\,\epsilon_n$.

THEOREM 10. *If $\rho(x)$ is a non-negative summable function such that $[\rho(x)]^{-r}$ is summable, $r > 0$, if $T_n(x)$ for each n is a trigonometric sum of the nth order minimizing* (3), *and if sums $t_n(x)$ exist so that*

$$|\,f(x) - t_n(x)\,| \leqq \epsilon_n,$$

there is a constant C, independent of x and n, such that

$$|\,f(x) - T_n(x)\,| \leqq C n^{(r+1)/(mr)} \epsilon_n.$$

The sums $T_n(x)$ will converge uniformly toward $f(x)$ if sums $t_n(x)$ exist for which

$$\lim_{n\to\infty} n^{(r+1)/(mr)} \epsilon_n = 0.$$

The last condition will be satisfied if $s=mr/(r+1)>1$ and the function $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0}\omega(\delta)/\delta^{1/s}=0$, or if $m>1$, $r=1/(m-1)$, and $f(x)$ has a continuous derivative. When $s<1$ a corresponding statement would involve further hypotheses on $f'(x)$, or, on occasion, higher derivatives of $f(x)$.

It is to be noted that Theorem 10, like the other even-numbered theorems throughout the paper, *applies even at points where* $\rho(x)$ *vanishes.*

7. *Convergence of Polynomial Approximation over Entire Interval.* The theorems that are still to be enumerated can be proved by combination of devices that have been employed above, and may be dismissed with a brief summary.

Throughout the remainder of the paper it will be understood that $f(x)$ is a given function continuous for $a\leqq x\leqq b$, and that $\rho(x)$ is non-negative and summable over (a, b). Whenever the symbol ϵ_n is used, it is with the implication that there exists a polynomial $p_n(x)$, of the nth degree, such that

$$|f(x)-p_n(x)|\leqq\epsilon_n$$

for $a\leqq x\leqq b$.

A proof related to that of Lemma 5 as the proof of Lemma 2 is related to that of Lemma 1 gives the following result.*

LEMMA 6. *If* $P_n(x)$ *is a polynomial of the nth degree, and*

$$G_{ns}=\int_a^b|f(x)-P_n(x)|^s dx,$$

then

$$|f(x)-P_n(x)|\leqq 4[4n^2G_{ns}/(b-a)]^{1/s}+5\epsilon_n$$

for $a\leqq x\leqq b$.

In deriving the following theorems, if the interval (c, d) is not the whole of (a, b), the Lemma is to be restated with reference to the interval (c, d) for the purposes of the demonstration:

THEOREM 11.† *If* $P_n(x)$ *is a polynomial of the nth degree minimizing the integral*

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* See Colloquium, p. 97.

† See Colloquium, p. 98.

$$(4) \qquad \int_a^b \rho(x)\,|\,f(x) - P_n(x)\,|^m dx, \qquad (m > 0),$$

and if $\rho(x)$ *has a positive lower bound for* $c \leqq x \leqq d$, *where* $a \leqq c < d \leqq b$, *there is a constant* C, *independent of* x *and* n, *such that*

$$|\,f(x) - P_n(x)\,| \leqq Cn^{2/m}\epsilon_n$$

for $c \leqq x \leqq d$.

THEOREM 12. *If* $P_n(x)$ *is a polynomial of the nth degree minimizing the integral* (4), *and if* $[\rho(x)]^{-r}$ *is summable over* (c, d), *with* $r > 0$, *where* $a \leqq c < d \leqq b$, *there is a constant* C, *independent of* x *and* n, *such that*

$$|\,f(x) - P_n(x)\,| \leqq Cn^{2(r+1)/(mr)}\epsilon_n$$

for $c \leqq x \leqq d$.

A sufficient condition for convergence, in terms of the order of magnitude of ϵ_n, is obvious in each case.

8. *Convergence of Polynomial Approximation over Entire Interval by Trigonometric Substitution.* The substitution $x = \cos\theta$ can be employed as in the proof of Lemma 3 to express an integral over the interval $(-1, 1)$ in terms of periodic functions; if $f(x)$ can be approximately represented by a polynomial $p_n(x)$ for $-1 \leqq x \leqq 1$ with an error not exceeding ϵ_n, the even periodic function $f(\cos\theta)$ is represented by the trigonometric sum $p_n(\cos\theta)$ so that

$$|\,f(\cos\theta) - p_n(\cos\theta)\,| \leqq \epsilon_n$$

for all values of θ. The further use of a linear substitution relates the interval $(-1, 1)$ to an arbitrary interval (a, b). The resulting transformation of Lemma 5 reads as follows.

LEMMA 7. *If* $P_n(x)$ *is a polynomial of the nth degree, and*

$$G'_{ns} = \int_a^b [(b - x)(x - a)]^{-1/2}\,|\,f(x) - P_n(x)\,|^s dx,$$

then

$$|\,f(x) - P_n(x)\,| \leqq 4(2nG'_{ns})^{1/s} + 5\epsilon_n$$

for $a \leqq x \leqq b$.

From this Lemma, restated for an interval (c, d), may be deduced theorems with regard to polynomials of closest approximation.

THEOREM 13. *If $P_n(x)$ is a polynomial of the nth degree minimizing the integral* (4), *and if $\rho(x)[(d-x)(x-c)]^{1/2}$ has a positive lower bound over (c, d), where $a \leqq c < d \leqq b$, there is a constant C, independent of x and n, such that*

$$|f(x) - P_n(x)| \leqq C n^{1/m} \epsilon_n$$

for $c \leqq x \leqq d$.

THEOREM 14. *If $P_n(x)$ is a polynomial of the nth degree minimizing the integral* (4), *and if*

$$[\rho(x)]^{-r}[(d - x)(x - c)]^{-(r+1)/2}$$

is summable over (c, d), with $r > 0$ where $a \leqq c < d \leqq b$, there is a constant C, independent of x and n, such that

$$|f(x) - P_n(x)| \leqq C n^{(r+1)/(mr)} \epsilon_n$$

for $c \leqq x \leqq d$.

9. *Convergence of Polynomial Approximation over Interior of Interval.* Adaptation of the reasoning by which Lemma 4 was established leads to the following lemma.

LEMMA 8. *If $P_n(x)$ is a polynomial of the nth degree, if*

$$G_{ns} = \int_a^b |f(x) - P_n(x)|^s dx,$$

and if N is the smallest integer $\geqq 1/s$, then

$$|f(x) - P_n(x)| \leqq \frac{K_1[(n + N)G_{ns}]^{1/s} + K_2 \epsilon_n}{[(b - x)(x - a)]^{N/2}}$$

for $a < x < b$, where

$$K_1 = 4 \cdot 2^{1/s}[(b - a)/2]^{N-(1/s)}, \quad K_2 = 5[(b - a)/2]^N.$$

It should perhaps be noted explicitly, as an item in the proof, that if there is a polynomial $p_n(x)$ of the nth degree such that

$|f(x)-p_n(x)| \leqq \epsilon_n$ for $-1 \leqq x \leqq 1$, then $\sin^N \theta\, p_n(\cos\theta)$ is a trigonometric sum of order $n+N$ approximating $\sin^N \theta\, f(\cos\theta)$ so that

$$\left| \sin^N\theta\, f(\cos\theta) - \sin^N\theta\, p_n(\cos\theta) \right| \leqq \epsilon_n$$

for all values of θ.

The following theorem, obtained without the use of Hölder's inequality, is a material improvement over a corresponding result previously published.*

THEOREM 15. *If $P_n(x)$ is a polynomial of the nth degree minimizing* (4), *and if $\rho(x)$ has a positive lower bound for $c \leqq x \leqq d$, where $a \leqq c < d \leqq b$, then $|f(x)-P_n(x)|$ has an upper bound of the order of $n^{1/m}\epsilon_n$ throughout any closed interval interior to (c, d).*

A sufficient condition for convergence in the interior of the interval (c, d), when $m>1$, is that $f(x)$ have a modulus of continuity $\omega(\delta)$ throughout (a, b) such that $\lim_{\delta\to 0}\omega(\delta)/\delta^{1/m}=0$; when $m=1$ it is sufficient that $f(x)$ have a continuous derivative throughout (a, b).

Finally, an application of Hölder's inequality already needed for the proof of Theorem 12 gives the following result.

THEOREM 16. *If $P_n(x)$ is a polynomial of the nth degree minimizing* (4), *and if $[\rho(x)]^{-r}$ is summable over (c, d), with $r>0$, where $a \leqq c < d \leqq b$, then $|f(x)-P_n(x)|$ has an upper bound of the order of $n^{(r+1)/(mr)}\epsilon_n$ throughout any closed interval interior to (c, d).*

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* See Colloquium, p. 101.