ON THE DEVELOPMENT OF FUNCTIONS IN SERIES OF ORTHOGONAL POLYNOMIALS*

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Introduction. A function $p(x)$, called the *weight* or *characteristic function*, non-negative on a given interval (a, b), finite or infinite, and such that all *moments*†

$$(1) \qquad \alpha_n = \int p(x)x^n dx, \qquad (n = 0, 1, 2, \cdots),$$

exist, with $\alpha_0 > 0$, gives rise, as is known [1],‡ to a system of orthogonal and normal Tchebycheff polynomials (*OP*)

$$(2) \quad \phi_n(x; p) \equiv \phi_n(x) = a_n x^n + \cdots, \qquad (n = 0, 1, \cdots; a_n > 0),$$

uniquely determined by the following relations:

$$(3) \qquad \int p(x)\phi_m(x)\phi_n(x)dx = \delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases}$$

$(m, n = 0, 1, \cdots)$. If $G_s(x) = \sum_{i=0}^{s} g_i x^i$ denotes an *arbitrary* polynomial of degree $\leqq s$ (subject in some cases to certain explicitly stated conditions), (2) yields

$$(4) \qquad \int p(x)\phi_n(x)G_{n-1}(x)dx = 0, \qquad (n = 1, 2, \cdots).$$

The most important and best known *OP* are the so-called "classical" polynomials of

$$(5) \quad \begin{aligned} &\text{(J) Jacobi: } (a, b) \text{ finite, say, } (-1, 1); \\ &\qquad\qquad p(x) = (1 + x)^{\alpha-1}(1 - x)^{\beta-1}, \ (\alpha, \beta > 0). \\ &\text{(L) Laguerre: } (a, b) = (0, \infty);\ p(x) = x^{\alpha-1}e^{-x}, \qquad (\alpha > 0). \\ &\text{(H) Hermite: } (a, b) = (-\infty, \infty);\ p(x) = e^{-x^2}. \end{aligned}$$

We note the following important cases of (J):

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† Here and hereafter $\int$ means $\int_a^b$.

‡ Numbers in brackets refer to the Bibliography at the end.

Legendre polynomials ($\alpha=\beta=0$), trigonometric polynomials ($\alpha=\beta=1/2$).

The *OP* have many interesting and increasingly important applications in pure analysis, as well as in mathematical physics, mathematical statistics, and mechanics. We may mention approximate evaluation of definite integrals (mechanical quadratures), interpolation, curve fitting, certain oscillation problems in engineering mechanics, and investigation of certain classes of polynomials (for example, monotonic in (a, b)). The most important of such applications, which served to introduce the general *OP* in analysis, is their use in expanding *arbitrary* functions in series. In fact, making use of (3), we get the formal expansion

$$(6) \qquad f(x) \sim \sum_{n=0}^{\infty} f_n \phi_n(x), \qquad f_n = \int p(x) f(x) \phi_n(x) dx,$$

of any function $f(x)$, for which the integrals expressing the f_n exist. We see that (6) is built up in the same manner as the trigonometric expansion, the ordinary Fourier series. The latter employs the simplest orthogonal sequence $\{\sin nx, \cos nx\}$, $(n=0, 1, \cdots)$, while (6) employs orthogonal polynomials, which may be considered the next simplest orthogonal functions.

A fundamental question naturally arising in connection with the expansion (6) is that of its convergence. *Given the sequence* $\{\phi_n(x)\}$, *for what classes of functions* $f(x)$ *does* (6) *converge in* (a, b) *or in a part thereof?* The present paper is devoted to a discussion of various methods used in dealing with the convergence properties of the expansion (6). It is confined to a single real variable and to ordinary convergence. The proofs are in general but briefly sketched, if not omitted, our aim being to bring out general ideas. The topics treated can be classified as follows.

I. General properties of the expansion (6).

II. General methods for investigating the convergence of (6).

(*a*) Case of continuous functions; use of Weierstrass' theorem.

(*b*) Application of the theory of integral equations.

(*c*) Application of the general theory of orthogonal functions; Rademacher-Menchoff theorem, Lebesgue constants.

(*d*) The use of the asymptotic expression of $\phi_n(x)$, $(n\rightarrow\infty)$ (Dirichlet integral); equiconvergence.

III. Special methods for investigating the convergence of (6).
(*a*) From summability to convergence.
(*b*) Use of the closure equation.
(*c*) Use of the differential equation (for the classical *OP*).
(*d*) Comparison method.

These methods naturally vary in generality and power. This is best illustrated by applying the various methods to expansions in series of Legendre polynomials, the oldest and the best known *OP*. A comparison of the criteria of convergence thus obtained brings out the following fact: the more general the method, the less does it utilize the special character of the functions involved; hence, a gain in generality is accompanied, as a rule, by a loss in the preciseness of the results obtained.

The supreme goal, when dealing with (6), is to show that, regarding convergence, it behaves in a certain subinterval of (a, b) like the ordinary Fourier series expansion of $f(x)$ or of some function simply related to it. This constitutes what we call the *equiconvergence theorem*, of greatest importance in our theory. Indeed, we well know the wide range of validity of the Fourier series expansion, which makes it such a powerful analytical instrument. The classes of *OP* for which this goal is attainable are evidently the most interesting, and the method by which the goal has been attained is evidently the most powerful one.

I. Some General Properties of the Expansion (6).

1. *Various Representations of the Remainder in* (6). We rewrite (6) in the form

$$(7)\qquad f(x) = \sum_{i=0}^{n} f_i\phi_i(x) + R_n(x; f) \equiv S_n(x; f) + R_n(x; f),$$

whence, by use of (3) and (4), it follows that $R_n(x; G_n) \equiv 0$,

$$(8)\qquad \begin{aligned} 1 &= \int p(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy \equiv \int p(y)K_n(x, y)dy, \\ S_n(x; f) &= \int p(y)f(y)K_n(x, y)dy, \end{aligned}$$

and, by use of Darboux's formula [2],

$$
\begin{aligned}
K_n(x, y) &\equiv \sum_{i=0}^{n} \phi_i(x)\phi_i(y) \\
&\equiv \frac{a_n}{a_{n+1}}\left[\frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y}\right], \\
K_n(x, x) = K_n(x) &= \frac{a_n}{a_{n+1}}\left[\phi_{n+1}'(x)\phi_n(x) - \phi_n'(x)\phi_{n+1}(x)\right] \\
&= \sum_{i=0}^{n} \phi_i^2(x).
\end{aligned} \tag{9}
$$

$$
S_n(x; f) = \frac{a_n}{a_{n+1}}\int p(y)f(y)\frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y}dy, \tag{10}
$$

$$
R_n(x) \equiv R_n(x; f) = \int p(y)[f(x) - f(y)]K_n(x, y)dy \tag{11}
$$

$$
= \frac{a_n}{a_{n+1}}\int p(y)[f(x) - f(y)]\frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y}dy,
$$

$$
\begin{aligned}
R_n(x; f) &= F(x) - \int p(y)[F(x) - F(y)]K_n(x, y)dy, \\
F(x) &\equiv f(x) - G_n(x),
\end{aligned} \tag{12}
$$

2. *Bessel Inequality. Closure.* The general assumption will be now made that $f(x)$ is of the class L_p^2, that is, $p(x)f^2(x)$ is L-integrable in (a, b). (Similarly, $f(x)$ being of the class L_p means $p(x)f(x)$ is L-integrable.) A ready application of (3) yields

$$
\int p(y)\left[f(y) - \sum_{i=0}^{n} f_i\phi_i(y)\right]^2 dy = \int p(y)f^2(y)dy - \sum_{i=0}^{n} f_i^2,
$$

which leads to the *Bessel inequality*:

$$
\begin{aligned}
\int p(y)[f(y) - S_n(y; f)]^2 dy &= \int p(y)R_n^2(y)dy \\
&= \int p(y)f^2(y)dy - \sum_{i=0}^{n} f_i^2 \geqq 0, \qquad (n = 0, 1, \cdots).
\end{aligned} \tag{13}
$$

Hence we may write

$$
\sum_{n=0}^{\infty} f_n^2 \text{ converges and is } \leqq \int p(y)f^2(y)dy. \tag{14}
$$

If in (14) *we have the right to use the equality sign, then we have the so-called Parseval formula* (*closure equation*), one of the most important in the theory of orthogonal functions:

$$(15) \qquad \sum_{n=0}^{\infty} f_n^2 = \int p(y)f^2(y)dy,$$

where $f(x)$ is of class L_p^2, and $f_n = \int p(x)f(x)\phi_n(x)dx$. This, in connection with (13), leads to the equation

$$(16) \qquad \int p(y)R_n^2(y)dy = \sum_{i=n+1}^{\infty} f_i^2 = o(1), \qquad (n \to \infty).$$

In the theory of *OP* it is shown: (i) *Parseval's formula always holds for* (a, b) *finite*; (ii) for (a, b) infinite its validity is intimately connected with the character, determined or indeterminate, of the *moment-problem* related to $p(x)$, that is, the problem whether the system of infinitely many equations

$$\int x^n d\psi(x) = \int p(x)x^n dx = \alpha_n, \qquad (n = 0, 1, \cdots),$$

where the unknown function $\psi(x)$ is monotonic non-decreasing in (a, b) with $\psi(a) = 0$ (and the left-hand integral is a Stieltjes integral), has or has not solutions distinct from the given one: $\psi(x) = \int_a^x p(x)dx$.* In particular, *Parseval's formula holds for the polynomials of Laguerre and Hermite. The validity of Parseval's formula is assumed throughout the subsequent discussion.*

3. *Consequences Derived from Parseval's Formula.* We know that

$$(17) \qquad \lim_{n\to\infty} f_n = 0,$$

for any $f(x)$ of the class L_p^2. The set of relations

$$(18) \qquad \int p(x)f(x)\phi_n(x)dx = 0, \qquad (n = 0, 1, \cdots),$$

implies $(p(x))^{1/2}f(x) = 0$ almost everywhere in (a, b).† In other words, two functions having the same *Fourier coefficients*

* The sharp distinction between the cases of (a, b) finite and infinite, as stated above, is clearly seen from the following example due to Stieltjes [3]:

$$\int_0^\infty e^{-x^{1/4}} \sin (x^{1/4})\, x^n dx = 0, \qquad (n = 0, 1, 2, \cdots).$$

† *If* (a, b) *is finite,* (18) *is valid for any* $f(x)$ *of the class* L_p *in* (a, b). In fact,

in (6) are equivalent, that is, their difference multiplied by $(p(x))^{1/2}$ vanishes almost everywhere in (a, b).*

The product

$$(p(x))^{1/2}S_n(x;f) \tag{19}$$

converges, for $n \to \infty$, *on the average* to $(p(x))^{1/2}f(x)$ on (a, b) [4]. It follows that a subsequence $(p(x))^{1/2}S_{n_k}(x;f)$ can be extracted which converges to $(p(x))^{1/2}f(x)$ in the ordinary sense $(n_k \to \infty)$ almost everywhere in (a, b). Hence, if $S_n(x;f)$ converges, for $n \to \infty$, on a set $E \subset (a, b)$ of positive measure (where we assume $p(x) \neq 0$), it necessarily converges to the value $f(x)$ almost everywhere on E.

We have also

$$\int p(x)f(x)F(x)dx = \sum_{n=0}^{\infty} f_n F_n, \tag{20}$$

where F is of class L_p^2, and

$$F_n = \int p(x)F(x)\phi_n(x)dx.$$

This is readily obtained by applying (15) to $f(x) \pm F(x)$. Assume further that

$$\int_c^d \frac{dx}{p(x)} \quad \text{exists}, \qquad (c, d) \subset (a, b). \tag{21}$$

Take in (20), $F(x) = 1/p(x)$ in (c, d), and $F(x) = 0$ elsewhere in (a, b); then $F_n = \int_c^d \phi_n(x)dx$, and we thus obtain the following important result:

$$\int_c^d f(x)dx = \sum_{n=0}^{\infty} f_n \int_c^d \phi_n(x)dx, \tag{22}$$

which tells us that *the expansion* (6), *whether convergent or not, can be integrated term by term in any interval* (c, d) *which is part*

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reduce (a, b) to $(-1, 1)$. Put $x = \cos\theta$ and denote $p(\cos\theta)f(\cos\theta)$ by $F(\theta)$. Then $\int_{-1}^{1} p(x)f(x)\phi_n(x)dx = 0$, $(n = 0, 1, \cdots)$, is equivalent to $\int_{-1}^{1} p(x)f(x)x^n dx = 0$ or to $\int_0^\pi F(\theta)\cos n\theta d\theta = 0$, $(n = 0, 1, \cdots)$. Define $F_1(\theta) = F(\theta)$ in $(0, \pi)$, and $F_1(\theta) = -F(-\theta)$ in $(-\pi, 0)$. Then $\int_{-\pi}^{\pi} F_1(\theta) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} n\theta d\theta = 0$ for all $n \geqq 0$; hence, by the theory of trigonometric series, $F_1(\theta) = 0$ almost everywhere in $(-\pi, \pi)$.

* Stieljes [3].

of (a, b), *where* (21) *is satisfied*,* *the convergence in* (22) *being uniform if d varies.*

We now turn again to the remainder $R_n(x;f)$ in (6) and assume, for definiteness, x to be a fixed point inside (a, b). Rewrite (11) as follows:

$$R_n(x;f) = \frac{a_n}{a_{n+1}}\left(\int_a^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^{b}\right) p(y)[f(x) - f(y)]$$

$$\cdot K_n(x, y)dy \equiv i_1 + i_2 + i_3$$

$$(23) \qquad = \frac{a_n}{a_{n+1}}\left\{\phi_{n+1}(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy\right.$$

$$\left. - \phi_n(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_{n+1}(y)dy\right\} + i_2 + i_3.$$

(Here and hereafter ϵ denotes a sufficiently small positive quantity, properly chosen.) In i_1 (similar considerations apply to i_3) define

$$\psi(y) = \frac{f(x) - f(y)}{x - y} \text{ in } (a, x - \epsilon), \qquad \psi(y) = 0 \text{ in } (x + \epsilon, b):$$

Then

$$\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = \int_a^b p(y)\psi(y)\phi_n(y)dy,$$

and we thus conclude, by (17), that

$$(24) \qquad \lim_{n\to\infty}\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = 0.$$

Furthermore, if (a, b) is finite, then a_n/a_{n+1} is bounded [1]: $a_n/a_{n+1} = O(1)$. If, in addition,

$$(25) \qquad \phi_n(x) = O(1)$$

at the given point x, then

$$(26) \qquad \lim_{n\to\infty}\frac{a_n}{a_{n+1}}\phi_{n+1}(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = 0.$$

It follows that the behavior of $R_n(x; f)$ depends here solely

* For Legendre polynomials (22) holds for any $f(x)$ of the class L [5].

upon its component i_2. The important conclusion resulting from the above considerations can be stated as follows. *For* (a, b) *finite, the convergence of the expansion* (6) *to* $f(x)$ *at a given point* x *where the sequence* $\{\phi_n(x)\}$ *remains bounded for* $n\to\infty$ *depends solely upon the nature of* $f(x)$ *in the immediate neighborhood of* x [6].

In other words, if two functions $f_1(x)$ and $f_2(x)$ coincide in an arbitrarily small neighborhood $(x-\epsilon, x+\epsilon)$ of the point x, then

$$(27)\qquad \begin{aligned} &\lim\ [R_n(x; f_1) - R_n(x; f_2)] = 0, \\ &\lim\ [S_n(x; f_1) - S_n(x; f_2)] = 0, \end{aligned} \qquad (n\to\infty),$$

for $R_n(x; f_1-f_2)$ is here reduced to the components i_1 and i_3 in (23), written for $f_1(x)-f_2(x)$.

We now add the further condition that $f(x)$ *satisfies a Lipschitz condition* in the neighborhood of the point x, that is,

$$(28)\qquad |f(y') - f(y'')| \leqq \lambda |y' - y''|,$$

$$(x - \epsilon \leqq y', y'' \leqq x + \epsilon;\ \lambda = \text{const., independent of } y', y'').$$

Define $F(y)$ as follows:

$$F(y) = \frac{f(x) - f(y)}{x - y} \text{ in } (x - \epsilon, x + \epsilon),\ y \neq x,$$

$$= 0 \text{ elsewhere in } (a, b), \text{ and for } y = x.$$

Under condition (28), $F(y)$ is seen to be of the class L_p^2, and by the same reasoning as applied above to $i_{1,3}$, we show that here $\lim i_2 = 0$, so that $\lim R_n(x; f) = 0$, $(n\to\infty)$. The same conclusion holds if

$$(29)\qquad |f(y') - f(y'')| \leqq \lambda |y' - y''|^{\alpha}, \text{ with } \alpha > 1/2,$$

where λ, y', y'' are as in (28) (*Lipschitz condition of order* α), *provided* $p(x)$ *is bounded in* $(x-\epsilon, x+\epsilon)$. Thus we have found, as an immediate consequence of Parseval's formula, that *the expansion* (6) *converges to* $f(x)$ *at any point* x *in the finite interval* (a, b) *in the neighborhood of which* $f(x)$ *satisfies a Lipschitz condition of order* 1 (*or of order* $>1/2$, *if* $p(x)$ *is bounded in this neighborhood*), *provided the sequence* $\{\phi_n(x)\}$ *is bounded at the point* x *under consideration.*

The latter condition is satisfied for various classes of *OP*, often uniformly in $(a+\epsilon, b-\epsilon)$, as illustrated by Jacobi polynomials. (This is usually established by means of the asymptotic expression for $\phi_n(x)$.)

4. *Some Extremal Properties of the Expansion* (6). By means of (3) and (4) we readily derive the following relations:

$$\int p(y)\left[f(y) - \sum_{i=0}^{n} (f_i + h_i)\phi_i(y)\right]^2 dy$$
$$= \int p(y)f^2(y) - \sum_{i=0}^{n} f_i^2 + \sum_{i=0}^{n} h_i^2 ,$$

where the h_i denote arbitrary constants; whence, by (13),

$$(30) \quad \int p(y)[f(y) - S_n(y;f)]^2 dy \leqq \int p(y)[f(y) - G_n(y)]^2 dy,$$

where the equality sign holds only for $G_n(y) \equiv S_n(y;f)$. This is an important extremal property which led Tchebycheff to the introduction of the general *OP* in analysis, *considering* (6) *as furnishing interpolation in the sense of least squares.*

II. General Methods for Investigating the Convergence Properties of the Expansion (6).

1. *Case of Continuous Functions.* The convergence of (6) evidently depends upon the order of magnitude (with respect to n) of the chosen polynomials $\phi_n(x)$ and of the coefficient f_n. While the former may be investigated once for all, by studying the properties of $p(x)$, the latter essentially depends on the nature of $f(x)$. We proceed to show that the study of the order of f_n is readily achieved in the important case where $f(x)$ *is continuous in the finite interval* (a, b). Here one naturally thinks first of Weierstrass' approximation theorem: *any function continuous in a given finite interval can be therein approximated uniformly and indefinitely by means of polynomials of ever increasing degrees.* Moreover, Tchebycheff has shown, that *among all polynomials of degree* $\leqq n$, *there exists a unique polynomial* $\Pi_n(x;f)$ *of "best approximation"* $(=E_n(f))$ *to* $f(x)$ *on* (a, b), that is,

$$(31) \quad E_n(f) = \max |f(x) - \Pi_n(x;f)| \leqq \max |f(x) - G_n(x)|,$$

$(a \leqq x \leqq b)$, where, by Weierstrass' theorem,

$$(32) \quad \lim_{n\to\infty} E_n(f) = 0.$$

Here, as in many other problems dealing with continuous functions, Weierstrass' theorem proves a powerful and readily applicable tool.

(i) We have, first, by (4),

$$
(33)\qquad \begin{aligned} f_n &= \int p(y)[f(x) - G_{n-1}(x)]dx \\ &= \int p(x)[f(x) - \Pi_{n-1}(x; f)]\phi_n(x)dx. \end{aligned}
$$

Hence, applying Schwartz's inequality and making use of (31), we find that

$$
(34)\qquad | f_n\phi_n(x) | \leqq (\alpha_0)^{1/2}E_{n-1}(f) | \phi_n(x) | .
$$

While this estimate, resulting from very simple considerations, is rather crude, it has the advantage of showing in many cases the convergence of (6) directly, without any further discussion. In fact, the order of magnitude of $E_n(f)$, depending upon the continuity properties of $f(x)$, is well known, thanks to the work of Lebesgue, de la Vallée-Poussin, S. Bernstein, and Dunham Jackson. Thus

$$
| f^{(p)}(x') - f^{(p)}(x'') | \leqq \lambda | x' - x'' |^\alpha, \quad (a \leqq x', x'' \leqq b; 0 < \alpha \leqq 1),
$$

implies $E_n(f) = O(n^{-p-\alpha})$, $(p \geqq 0)$;

(35) $f^{(p)}(x)$ continuous in (a, b) implies $E_n(f) = o(n^{-p})$;

$$
| f(x+\delta) - f(x) | \cdot | \log | \delta | |_{\delta \to 0} \to 0 \text{ (Dini–Lipschitz condition)}
$$

implies $E_n(f) = o(1/\log n)$.

If now the order of $\phi_n(x)$ is known, say

$$
(36)\qquad | \phi_n(x) | = O(n^\sigma)
$$

for a certain x, then, for the same x,

$$
(37)\qquad | f_n\phi_n(x) | = O(n^\sigma E_{n-1}(f)).
$$

This, combined with (35), enables us to indicate at once classes of continuous functions for which (6) converges for the above x. If, for example, $\sigma = 0, 1$, then (6) certainly converges (absolutely) for the above values of x, if the first or the second derivative of $f(x)$, respectively, satisfies in (a, b) a Lipschitz condition of an arbitrary positive order α, for then

$$|f_n\phi_n(x)| = O(n^{-1-\alpha}), \qquad (\alpha > 0).$$

The said convergence is uniform in any interval $\subset (a, b)$ in which $|\phi_n(x)| = O(n^\sigma)$, with $\sigma = 0, 1$.

(ii) More refined results are obtained from (12), with $G_n(x) \equiv \Pi_n(x; f)$. We thus get, applying Schwartz's inequality to the integral on the right, and using (3) once more,

$$(38) \quad |R_n(x; f)| \leqq E_n(f)\{1 + (K_n(x))^{1/2}\} = O(E_n(f)(K_n(x))^{1/2}).$$

$$(39) \quad |R_n(x; f)| = O(E_n(f)n^{\sigma+1/2}), \text{ if } |\phi_n(x)| = O(n^\sigma),$$

where $\sigma > -1$. We recall that $a_n/a_{n+1} = O(1)$ for (a, b) finite.

(iii) Still more refined results can be obtained [7, 8], if we again use (12), breaking up the integral on the right somewhat along the lines of (23):

$$\int_a^b = \int_a^{c+\epsilon} + \int_{c+\epsilon}^{x-\epsilon_n} + \int_{x-\epsilon_n}^{x+\epsilon_n} + \int_{x+\epsilon_n}^{x+\epsilon} + \int_{x+\epsilon}^{b}$$
$$\equiv i_1 + i_2 + i_3 + i_4 + i_5,$$
$$(a < c \leqq x \leqq d < b;\ \epsilon_n = o(1)(n \to \infty),\ |\phi_n(x)| = O(n^\sigma),\ \sigma > -1).$$

In $i_{1,5}$, where $|x - y| \geqq \epsilon$, we use Darboux's formula (9) for $K_n(x, y)$ and get integrals of the type $\int p(y)|\phi_n(y)|dy$ which $= O(1)$ (by Schwartz's inequality), so that

$$|i_{1,5}| = O(n^\sigma E_n(f)).$$

In $i_{2,4}$ put $x - y = u$; then

$$|i_{2,4}| = O(n^{2\sigma}E_n(f)) \int_{\epsilon_n}^{b-a} \frac{du}{u} = O(E_n(f) \cdot n^{2\sigma} |\log|\epsilon_n||).$$

Finally, in i_3 use $K_n(y) = O(n^{2\sigma+1})$, so that

$$|i_3| = O\left(E_n(f) \cdot n^{2\sigma+1} \int_{x-\epsilon_n}^{x+\epsilon_n} p(y)dy\right).$$

By taking $\epsilon_n = n^{-\beta}$, with a properly chosen $\beta > 0$, we can find the order of $R_n(x; f)$ and the class of continuous functions for which $\lim_{n\to\infty} R_n(x; f) = 0$.

NOTE. In the case under discussion, by (30) and (16),

$$(40)\quad \int p(y)R_n^2(y)dy \leqq \int p(y)\,|f(y) - \Pi_n(y;f)|^2 dy \leqq \alpha_0 E_n^2(f),$$

$$(41)\qquad E_n(f) \geqq \frac{1}{\alpha_0^{1/2}}\left(\sum_{i=n+1}^{\infty} f_i^2\right)^{1/2}.$$

A lower bound for $E_n(f)$ is of importance in the theory of approximation. The relations (40) and (41), for a suitable orthogonal system $\{\phi_n(x)\}$, furnish such a lower bound, very acceptable for certain classes of functions [9].

2. *Application of the Theory of Integral Equations.* Rewrite (3) and (6), respectively, as

$$(42)\qquad \int \Phi_m(x)\Phi_n(x)dx = \delta_{mn}, \qquad (m, n = 0, 1, \cdots),$$

$$(43)\qquad F(x) \sim \sum_{n=0}^{\infty} f_n \Phi_n(x),$$

where

$$\Phi_n(x) = (p(x))^{1/2}\phi_n(x);\ F(x) = f(x)(p(x))^{1/2};\ f_n = \int F(x)\Phi_n(x)dx.$$

Consider the formal expression

$$(44)\quad K(x, y) \equiv \sum_{n=0}^{\infty} \frac{\Phi_n(x)\Phi_n(y)}{l_n} = (p(x)p(y))^{1/2}\sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{l_n},$$

where l_n denote constants. For finite (a, b) the positive constants l_n in (44) may be chosen in infinitely many ways so that its right-hand member converges absolutely and uniformly in (a, b). Take, for example,

$$l_n = M_n^2 n^{1+\alpha},\ (\alpha > 0), \qquad M_n = \max |\phi_n(x)| \text{ in } (a, b).$$

The same holds for (a, b) infinite, if $(p(x))^{1/2}|\phi_n(x)|$ does not exceed in (a, b) a finite positive quantity M_n for each n (example: polynomials of Laguerre and Hermite). We have then, by virtue of orthogonality,

$$(45)\quad (p(x))^{1/2}\phi_n(x) = l_n \int (p(y))^{1/2}K(x, y)\phi_n(y)dy,$$

$$(46)\qquad \Phi_n(x) = l_n \int K(x, y)\Phi_n(y)dy, \qquad (n = 0, 1, \cdots).$$

Thus, in cases under consideration, the functions $\{\Phi_n(x)\}$ are the fundamental functions for the linear integral equation (46), and the expansion (43), which is but a modification of (6), may be studied by applying the general theory of linear integral equations, more precisely, the Hilbert-Schmidt expansion theorem, which in its simplest form can be stated as follows. *Let $K(x, y)$ be a symmetric continuous kernel in the domain $a \leqq x$, $y \leqq b$, and let $\omega_n(x)$, $(n=1, 2, \cdots)$, denote the corresponding fundamental functions. Any function of the form*

$$f(x) = \int K(x, y)h(y)dy, \tag{47}$$

where $h(x)$ is square-integrable in (a, b), can be expanded in a series according to the $\omega_n(x)$, which converges absolutely and uniformly; the expansion is of the form

$$f(x) = \sum_{n=1}^{\infty} \frac{h_n}{l_n} \omega_n(x), \tag{48}$$

where

$$h_n = \int h(y)\omega_n(y)dy ,$$

and where l_n denote characteristic numbers.

However, we encounter here two serious difficulties. First, the actual construction of the kernel $K(x, y)$ in (45) and (46), depending largely upon the choice of the constants l_n in (44), is not so easy to carry out even in the simplest cases of Legendre and Hermite polynomials [10, 11, 12]; secondly, the criteria for convergence of (43) (or (6)) thus obtained are far too strict, for the method, due to its very generality, does not utilize sufficiently the individual properties of the orthogonal functions involved. Thus Weyl [11], applying the theory of integral equations to the expansion of a given function $f(x)$ according to Hermite functions $\{e^{-x^2/4}\phi_n(x)\}$ orthogonal in $(-\infty, \infty)$, finds the said expansion converges if $f(x)$ and $f'(x)$ are continuous and both integrals $\int_{-\infty}^{\infty} x^2 f^2(x)dx$, $\int_{-\infty}^{\infty} f'^2(x)dx$ exist. These criteria are far inferior to those obtained by other methods [13, 14]. Even more stringent are the criteria of W. Lebedeff [12] obtained by means of a different kernel. The same remark applies to the discussion by the said writers of expansions in series of Laguerre polynomials.

A different application of the theory of integral equations to *OP* has been given by N. Kryloff [15]. His main point is the following generalized Schmidt's Lemma.

Consider a sequence $\{\omega_n(x)\}$, $(n=0, 1, \cdots)$, *of class* L^2, *orthogonal and normal in* (a, b), *for which Parseval's formula holds, that is, for any* $f(x)$ *of class* L^2,

$$\int f^2(x)dx = \sum_{n=0}^{\infty} f_n^2, \qquad f_n = \int f(x)\omega_n(x)dx.$$

If $\psi(x)$ *is of class* L^2 *and* $F(x, z)$ *is such that* $\int F^2(x, z)dz \leqq M$ *(independent of* z*) for any* z *in* (z_0, z_1), *then*

$$\int F(x, z)\psi(x)dx = \sum_{n=0}^{\infty} \int F(x, z)\omega_n(x)dx \cdot \int \psi(x)\omega_n(x)dx, \tag{49}$$

and the right-hand member converges absolutely and uniformly for $z_0 \leqq z \leqq z_1$. (See the Hilbert-Schmidt theorem given above.)

Suppose now we wish to investigate the convergence of the expansion

$$f(x) = \sum_{n=0}^{\infty} f_n\phi_n(x), \qquad f_n = \int p(x)f(x)\phi_n(x)dx. \tag{6}$$

The ingenious idea of Kryloff is to *identify* (6) *with* (49), *and then to apply* the preceding Lemma. Choose, first, $F(x, z)$ and the sequence $\{\omega_n(x)\}$, then $\psi(x)$, so that

$$\begin{aligned} &\int F(x, z)\omega_n(x)dx = \phi_n(z), && (a \leqq z \leqq b), \\ &\int \psi(x)\omega_n(x)dx = \int p(x)f(x)\phi_n(x)dx, && (n = 0, 1, \cdots). \end{aligned} \tag{50}$$

If such a choice is possible, the right-hand member in (6) (where x is replaced by z) does become identical with that in (49), namely,

$$\sum_{n=0}^{\infty} f_n\phi_n(z) \equiv \sum_{n=0}^{\infty} \int \psi(x)\omega_n(x)dx \cdot \int F(x, z)\omega_n(x)dx.$$

Hence it converges absolutely and uniformly in (a, b) to the value of $f(x)$ (by virtue of (19)), if $F(x, z)$ satisfies the condition of the Lemma. Kryloff applies this method to Jacobi polynomials in $(-1, 1)$ and shows, on the basis of their differential equation (see below, (85)), that such a choice as required by (50) is possible, if α, $\beta > 1$, and *the expansion* (6) *converges absolutely and uniformly in* $(-1, 1)$, *if* $f(x)$ *and* $f'(x)$ *are of class* L_p^2.

3. *Application of the General Theory of Orthogonal Functions.* This is suggested by the very form of the expansion (43). The application in question, in order to attain the utmost in effectiveness, should be coupled with the special properties of the *OP* involved.

(i) *Use of Rademacher-Menchoff theorem* [16, 17]. This general theorem deals with a sequence $\{\omega_n(x)\}$ of functions orthogonal and normal in the finite interval (a, b), and tells us that *the expansion* $\sum_{n=0}^{\infty} c_n \omega_n(x)$ *converges almost everywhere in* (a, b), *if* $\sum_{n=2}^{\infty} c_n^2 \log^2 n$ *converges.** ($\omega_n(x)$ is of class L^2.)

This theorem, applied to (43), shows at once that (43) *converges to* $(p(x))^{1/2} f(x)$ *almost everywhere in the finite interval* (a, b), *if the coefficients* f_n *are such that* $\sum_{n=2}^{\infty} f_n^2 \log^2 n$ *converges.* The latter condition is satisfied in one of the following cases:

1. $0 < p(x) < M$, (M finite), or $0 < p(x) < M/((x-a)(b-x))^{1/2}$, $f(x)$ is of bounded variation in (a, b) [18].

2. $p(x) > 0$; $|f(x') - f(x'')| < \text{const.}/|\log|x'-x''||^{1+\epsilon}$, $(a \leqq x', x'' \leqq b)$† [19]. The Rademacher-Menchoff theorem lays the emphasis on the coefficients f_n in the expansion (6).

(ii) *Use of Lebesgue constants.* Here the emphasis is placed on the orthogonal functions employed. Using the expressions (11) and (12) for the remainder $R_n(x; f)$ in the expansion (6), we are led to introduce the Lebesgue constants:

$$\rho_n(x) = \int p(y) \, | K_n(x, y) | \, dy,$$

$$\bar{\rho}_n(x) = \int \left| \sum_{i=0}^{n} \Phi_i(x) \Phi_i(y) \right| dy, \tag{51}$$

* Note the presence of the factor $\log^2 n$. From the mere convergence of $\sum_{n=2}^{\infty} c_n^2$ follows only (by the Riesz-Fischer theorem) the existence of a function $f(x)$ of class L^2 such that $c_n = \int f(x) \omega_n(x) dx$, $(n = 0, 1, \cdots)$.

† And, a fortiori, if $f(x)$ satisfies in (a, b) a Lipschitz condition of an arbitrarily given order α (>0).

where x is given in (a, b), $(n=0, 1, \cdots)$, and where $\rho_n = \max \rho_n(x)$, $\bar{\rho}_n = \max \bar{\rho}_n(x)$ for $a \leqq x \leqq b$. The symbol max here and everywhere means, when necessary, *upper bound*. Hence, if $f(x)$ is bounded in (a, b),

$$(52) \quad |R_n(x;f)| \leqq 2M\rho_n(x) \leqq 2M\rho_n, \quad (a \leqq x \leqq b;\ |f(x)| \leqq M),$$

by (11); and if $f(x)$ is continuous in the finite interval (a, b),

$$(53) \quad |R_n(x;f)| \leqq E_n(f)\{1 + \rho_n(x)\} \leqq E_n(f)(1 + \rho_n),$$

$(a \leqq x \leqq b)$, by (12), where $G_n(x) \equiv \Pi_n(x; f)$. Here $\rho_n(x)$ is the upper bound of $S_n(x; f)$ for all $f(x)$ such that $|f(x)| \leqq 1$ in (a, b). In fact, for such $f(x)$,

$$|S_n(x;f)| \leqq \int p(y)\,|K_n(x, y)|\,dy = \rho_n(x).$$

Moreover, this upper bound is actually attained by the function $f(y) = \operatorname{sgn} K_n(x, y)$.* Equation (53), by virtue of (32), shows at once that *if* $\{\rho_n(x)\}$ *is bounded, then* (6) *converges to the value* $f(x)$ *at the given point* x *for any continuous* $f(x)$, the said convergence being uniform in any interval $\subset (a, b)$ where the Lebesgue constants remain uniformly bounded (in x and n). On the other hand, if the sequence $\{\rho_n(x)\}$ is unbounded, it can be shown, following Haar [20], that there exist continuous functions whose expansions (6) diverge at the point x. In any case, if the order, with respect to n, of $\rho_n(x)$ or ρ_n is known, equation (53), where (a, b) is assumed to be finite, will show at a glance for what classes of continuous functions (6) converges, by means of (35). If, for example, $\rho_n(x) = O(\log n)$, then (6) converges to $f(x)$ at the given point x, provided $f(x)$ satisfies in (a, b) a Dini-Lipschitz condition. In this connection it was shown by Rademacher [16] that

$$(54) \qquad \bar{\rho}_n(x) = O(n^{1/2}(\log n)^{3/2+\epsilon})$$

almost everywhere in (a, b); hence, (43) *converges to* $(p(x))^{1/2}f(x)$ *almost everywhere in* (a, b) *if* $f(x)$ *satisfies in* (a, b) *a Lipschitz condition of order* $> 1/2$.

---

* Note that $\operatorname{sgn} a = +1, -1, 0$, corresponding to $a > 0$, < 0, $= 0$.

The above discussion clearly shows the importance of the so-called "zero-" or "kernel series" (série-noyau, Kogbetliantz):*

$$(55)\qquad \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(y) \equiv (p(x)p(y))^{1/2}\sum_{n=0}^{\infty} \phi_n(x)\phi_n(y).$$

To illustrate, assume that

$$(56)\quad \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y) = 0 \text{ for } x \neq y, \qquad (a \leqq c \leqq x,\ y \leqq d \leqq b),$$

uniformly if $|x-y| \geqq \epsilon$. Assume further, in order to simplify the discussion and to bring out more clearly the underlying ideas, that (56) holds in (a, b),† and let $f(x)$ be continuous at a certain interior point x (where $p(x) \neq 0$). We have then‡

$$S_n(x; f) = \left(\int_a^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^{b}\right) p(y)f(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy$$
$$\equiv i_1 + i_2 + i_3,$$

where $i_1 = o(1)$, $i_3 = o(1)$, by (56);

$$i_2 = f(x)\int_{x-\epsilon}^{x+\epsilon} p(y)\sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy$$
$$+ \int_{x-\epsilon}^{x+\epsilon} p(y)[f(y) - f(x)] \sum_{i=0}^{n}\phi_i(x)\phi_i(y)dy = i_2' + i_2'';$$

---

* A penetrating investigation of this and related zero-series by E. Kogbetliantz for various classes of *OP* has yielded in recent years extremely general results regarding convergence and summability of the expansion (6) (see, for example, [14]).

† Ordinarily (56) is satisfied when the series on the left is subjected to a certain process of summation, so that the conclusions drawn pertain not to ordinary convergence, but to summability. The latter, under additional hypotheses, may lead to ordinary convergence (see below, p. 72).

‡ The end points need special attention. Thus, if (a, b) is finite, we generally write $\int_a^{x-\epsilon} = \int_a^{a+\epsilon_n} + \int_{a+\epsilon_n}^{x-\epsilon}$, where $\epsilon_n = o(1)$ is properly chosen, and if (a, b) is infinite, say $a = -\infty$, then we write $\int_{-\infty}^{-G} + \int_{-G}^{x-\epsilon}$, where G (>0) is sufficiently large (or even depends on n and increases indefinitely with n). Similar remarks apply to b. In other words, we must properly specify the behavior of $f(x)$ near the end points.

$$i_2' = f(x)\left(\int_a^b - \int_a^{x-\epsilon} - \int_{x+\epsilon}^b\right)p(y)\sum_{i=0}^n \phi_i(x)\phi_i(y)dy$$

$$= f(x) + o(1) \qquad \text{(by (8) and (56))};$$

$$|i_2''| \leqq \eta(\epsilon)\left(\int_a^b + \int_a^{x-\epsilon} + \int_{x+\epsilon}^b\right)p(y)\left|\sum_{i=0}^n \phi_i(x)\phi_i(y)\right|dy$$

$$= \eta(\epsilon)(\rho_n(x) + o(1));$$

$(|f(y) - f(x)| \leqq \eta(\epsilon)$ for $x - \epsilon \leqq y \leqq x + \epsilon; \eta(\epsilon) \to 0$, as $\epsilon \to 0)$.

Thus, we again come across the Lebesgue constants, namely,

(57) $S_n(x;f) = f(x) + \eta_1(\epsilon)\rho_n(x) + o(1), \quad (\eta_1(\epsilon) \to 0, \text{ as } \epsilon \to 0),$

and this shows that $\lim_{n\to\infty} S_n(x;f) = f(x)$, *if the sequence* $\{\rho_n(x)\}$ *is bounded*, even though the continuity of $f(x)$ was here assumed at the point x only.* The important point here is that, due to (56),

(58) $$\rho_n(x) = \int_{x-\epsilon}^{x+\epsilon} p(y)\left|\sum_{i=0}^n \phi_i(x)\phi_i(y)\right|dy + o(1), \qquad (n \to \infty).$$

In general, the estimate of $\rho_n(x)$ requires the use of the asymptotic expression for $\phi_n(x)$. Another method based on a theorem of Fubini will be omitted [21].

4. *Making Use of the Asymptotic Expression of* $\phi_n(x)$ $(n \to \infty)$. One can foresee that this method will prove the most powerful, for the said asymptotic expression is, we may say, a synthesis of the most intimate properties of $\phi_n(x)$. The essential features of this method may be exhibited as follows. For various classes of *OP* the asymptotic expression of $\phi_n(x)$, for $n \to \infty$, has in the first approximation the following form:

(59) $$\phi_n(x) = A_n\left[\cos(n^q\psi + \omega_n) + \frac{B_n}{n^p}\right], \qquad (q, p > 0),$$

and this holds in a certain subinterval of (a, b), where A_n, ψ, ω_n, B_n are certain functions of x, with ω_n, A_n, B_n depending generally on n, and A_n, B_n remaining bounded for $n \to \infty$. Thus, for example, for Legendre polynomials [2, 22]:

---

* In case $f(x)$ is of bounded variation in the neighborhood of the given point x (the usual assumption made in dealing with (6)), the previous analysis, somewhat refined, would introduce the quantities $f(x \pm 0)$.

$$
\begin{aligned}
P_n(x) &= \left(\frac{2}{2n+1}\right)^{1/2}\phi_n(x,1) = \frac{4}{\pi}\cdot\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)} \\
&\cdot\frac{1}{(2\sin\phi)^{1/2}}\cdot\left\{\cos\left[\left(n+\frac{1}{2}\right)\phi-\frac{\pi}{4}\right]+\frac{1}{2}\cdot\frac{1}{2n+3}\right. \\
&\cdot\frac{\cos\left[\left(n+\frac{3}{2}\right)\phi-\frac{3\pi}{4}\right]}{2\sin\phi}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1\cdot 3}{(2n+3)(2n+5)} \\
&\left.\cdot\frac{\cos\left[\left(n+\frac{5}{2}\right)\phi-\frac{5\pi}{4}\right]}{(2\sin\phi)^2}+\cdots\right\} \\
&= \left(\frac{2}{n\pi\sin\phi}\right)^{1/2}\left\{\cos\left[\left(n+\frac{1}{2}\right)\phi-\frac{\pi}{4}\right]+\frac{B_n(\phi)}{n}\right\}, \\
&\qquad (-1+\epsilon\leqq x=\cos\phi\leqq 1-\epsilon).
\end{aligned}
\tag{60}
$$

Similarly, for Hermite polynomials [13]:

$$
\begin{aligned}
\phi_n(x) &= e^{x^2/2}\cdot\pi^{-1/2}\cdot\left(\frac{2}{n}\right)^{1/4}\cdot\left\{\cos\left(x(2n)^{1/2}+\frac{n\pi}{2}\right)\right. \\
&\left.+\left(\frac{x^3}{6}-\frac{x}{2}\right)\cdot(2n)^{-1/2}\cdot\sin\left(x(2n)^{1/2}+\frac{n\pi}{2}\right)+\frac{B_n(x)}{n}\right\},
\end{aligned}
\tag{61}
$$

where $|x|\leqq A$ is finite and arbitrarily fixed.

In the first place, we find, by means of Darboux's formulas (9), an asymptotic expression for $K_n(x, y)$ and $K_n(x)$, x being a certain given point belonging to the interval where (59) is valid. Next, again rewrite (10) in the form

$$
\begin{aligned}
S_n(x;f) &= \left(\int_a^{x-\epsilon}+\int_{x-\epsilon}^{x+\epsilon}+\int_{x+\epsilon}^{b}\right)p(y)f(y)K_n(x,y)dy \\
&\equiv i_1+i_2+i_3,
\end{aligned}
\tag{62}
$$

making the usual assumption that $f(x)$ is of bounded variation in the neighborhood of the point x. The most difficult part of the investigation, which requires great ingenuity (and often laborious computation) is to show that i_1 and i_3 tend to 0, as n tends to infinity, for the asymptotic expression (59) does not hold, as a rule, at the end points. Here we must specify the proper behavior of $f(x)$. The integrals i_1 and i_3 having been disposed of, we proceed to the relatively simple investigation of i_2. Using (59) and making a proper change of the variable of integration, we are led to a finite number of integrals of the form

$$\int_\alpha^\beta u(z) \frac{\sin}{\cos} mz dz, \qquad (\alpha, \beta \text{ finite or infinite}), \tag{63}$$

$$\int_0^k u(z) \frac{\sin mz}{\sin z} dz, \qquad \text{Dirichlet integral}, \tag{64}$$

$$\int_h^k u(z) \frac{\sin mz}{\sin z} dz, \qquad (0 < h < k \leqq \pi/2). \tag{65}$$

The behavior, for $m \to \infty$, of these integrals is well known from the theory of trigonometric series [4]. Thus, $(63) = o(1)$, if $|u(z)|$ is L-integrable in (α, β) (Riemann-Lebesgue theorem); $(64) \to (\pi/2)u(+0)$, if $u(z)$ is of bounded variation in $(0, \pi/2)$; $(65) \to o(1)$, for the same $u(z)$. These three limiting relations enable us to complete our investigation; (64) here furnishes the desired limit of $S_n(x; f)$, $(n \to \infty)$, in terms of $f(x \pm 0)$. The generality of the results obtained in this manner depends upon two factors: (i) the thoroughness of the study of the components $i_{1,3}$ in (62), by which we can avoid unnecessarily heavy restrictions to be imposed upon the differentiability and integrability properties of $f(x)$, also upon its behavior at the end points; (ii) the more or less comprehensive character of the asymptotic expression used, that is, its range of validity, order of magnitude, and nature of the remainder. Thus, Kogbetliantz [14] has obtained very general results on the convergence (and summability) of series according to Hermite or Laguerre polynomials, by using asymptotic expressions for $\phi_n(x)$ *valid over an interval which increases indefinitely with* n. Haar [23], making use of the second approximation in (59), was able to establish the most interesting result in the theory of Legendre polynomials—the so-called "equiconvergence theorem" (see Introduction): $f(x)$ *being of the class* L^2, *its expansion* (6) *in series of Legendre polynomials behaves, with regard to convergence or divergence at any interior point* $x = \cos\theta$, $(0 < \theta < \pi)$, *like the Fourier cosine-series expansion of* $f(\cos\theta)$, *that is, denoting by* $\sigma_n(\theta)$ *the partial sum of the latter series, we have*

$$\lim_{n\to\infty} [S_n(\cos\theta) - \sigma_n(\theta)] = 0, \qquad (0 < \theta < \pi).$$

Quite recently, Szegö [24] has developed for Jacobi polynomials with arbitrary α, $\beta(>0)$ (see (5)) asymptotic expressions in *terms of Bessel Functions*, instead of trigonometric functions as in (59)–(61). His asymptotic expressions have the advantage of *being valid uniformly in any left-hand neighborhood of the end point* $x=1$. Thus one of the difficulties mentioned above is obviated, and we obtain in a simple manner the asymptotic expression of the Lebesgue constants and of many other definite integrals involving Jacobi polynomials. The most important application which Szegö makes of his asymptotic expressions is to derive a general "equiconvergence theorem," which, in case of Legendre polynomials, holds even for a more general class of functions than that in Haar's theorem (see below, p. 78).

In order to illustrate the difficulties arising in case of an infinite interval, we proceed to sketch the ingenious method of Uspensky in dealing with Hermite polynomials [13]. We write here

$$S_n(x;f) = \left(\int_{-\infty}^{-G} + \int_{-G}^{G} + \int_{G}^{\infty}\right) e^{-y^2} f(y) K_n(x, y)dy$$
$$\equiv i_1 + i_2 + i_3, \qquad (G>0, \text{ sufficiently large}).$$

The integral i_2 is reduced to the Dirichlet integral as usual, and furnishes in the limit, for $n\to\infty$, $[f(x+0)+f(x-0)]/2$, ($f(x)$ being of bounded variation in the neighborhood of the given point x). We now turn to i_3 (the analysis is similar for i_1). First, by Schwartz's inequality,

$$(66)\qquad i_3^2 \leqq \int_G^\infty e^{-y^2} f^2(y)dy \cdot \int_G^\infty e^{-y^2} K_n^2(x, y)dy.$$

The first factor in (66) does not cause trouble under the liberal assumption

$$(67)\qquad \int_A^\infty e^{-y^2} f^2(y)dy, \quad \text{and also} \quad \int_{-\infty}^{-A} e^{-y^2} f^2(y)dy,$$

exist for some $A > 0$.

We proceed to show that the second factor→0 as $n\to\infty$. By (9) and (10),

$$\phi'_{n+1}(x)\phi_n(x) - \phi'_n(x)\phi_{n+1}(x) = \frac{(2n+2)^{1/2}}{2}\int_{-\infty}^{\infty} e^{-y^2}\beta_n^2(y)dy$$

$$(68)\qquad \equiv I_n = \left(\frac{2n+2}{2}\right)^{1/2}\left(\int_{-\infty}^{-G} + \int_{-G}^{G} + \int_{G}^{\infty}\right)$$

$$\equiv I'_n + I''_n + I'''_n;$$

$$\beta_n(x, y) = \frac{\phi'_{n+1}(x)\phi_n(y) - \phi'_n(x)\phi_{n+1}(y)}{x - y},$$

where x is fixed and finite. The ingenious method of Uspensky consists in first evaluating asymptotically I_n, as a whole, using on the left side of (68) the asymptotic expressions of $\phi_n(x)$, $\phi'_n(x)$ (the latter also being a Hermite polynomial), then evaluating asymptotically I''_n, by using the above asymptotic expressions *under the integral sign* (the interval of integration being finite). *By subtraction we find at once the order of magnitude of* I'_n, I'''_n, and this completes the discussion of i_3, *without any further restriction being imposed on* $f(x)$. Laguerre polynomials are treated in a similar way. The criteria of convergence of (6) thus obtained are very sensitive indeed.

III. Some Special Methods for Investigating the Convergence Properties of the Expansion (6).

1. *From Summability to Convergence.* It is well known [25] that the Césaro (C), Abel (A), or Euler (E) summability of an infinite series $\sum_{n=0}^{\infty} u_n \equiv U$ implies its ordinary convergence if u_n is properly restricted. Thus, (i) if U is (C, k) summable, and $u_n = O(1/n)$, then the series converges (Hardy); (ii) if U is A-summable, and $u_n = O(1/n)$, then the series converges (Littlewood). The existence of generating functions for the classical OP, that is,

$$\Phi(x, t) = \sum_{n=0}^{\infty} c_n\phi_n(x)t^n, \qquad (c_n = \text{const.}),$$

whose explicit expressions are known, suggests in the first place Abel-summability. This was carried out by Hille [26] for Laguerre polynomials, where

$$(69)\qquad \frac{e^{-xt}}{(1-t)^{\alpha}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Gamma(n+\alpha)}{\Gamma(n+1)}\right)^{1/2} \phi_n(x) t^n,$$

where $|t| < 1$, and where x is finite and arbitrary. Corresponding to the expansion (6), we construct the series

$$(70)\qquad F(x, t) = \sum_{n=0}^{\infty} f_n \phi_n(x) t^n, \qquad (x \geqq 0, \text{ fixed}).$$

Making use of the asymptotic expression for $\phi_n(x)$ and of the expression of f_n, we show, first, that $F(x, t)$, as a function of t, is holomorphic for $|t| < 1$; next, that for $t \to 1-0$, $F(x, t) \to f(x_0)$ or $[f(x_0+0)+f(x_0-0)]/2$, at every point x_0 of continuity or of discontinuity of the first kind, respectively; in other words, we show that (6) *is A-summable, if* $\int_0^\infty e^{-ax} x^{\alpha-1} |f(x)| dx$ *exists for every* $a > 1/2$. By Littlewood's theorem, (6) will converge at $x = x_0$ (to the above value), if $f(x)$ is such that

$$(71)\qquad f_n \phi_n(x_0) = O(n^{-1}).$$

For $x_0 > 0$ and $\alpha = 1$, $\phi_n(x_0) = O(n^{-1/4})$; hence (71) is certainly satisfied if

$$(72)\qquad f_n = O(n^{-3/4}),$$

which is shown to hold if $f(x)$ is subject to some further restrictions.

2. *Application of the Closure Equation* (15). We follow here, with considerable modification, the method developed by Stekloff [27] for the case where

$$(73)\qquad p(x) \geqq p_0 > 0,$$

$p'(x)$ exists and is bounded in the finite interval (a, b),

$$(74)\qquad f'(x) \text{ is of class } L_p^2 \text{ in } (a, b).$$

Write

$$(75)\quad f(x) = \sum_{i=0}^{n} f_i \phi_i(x) + R_n; \qquad f'(x) = \sum_{i=0}^{n} f_i \phi_i'(x) + R_n',$$

where R_n denotes $R_n(x; f)$, and where R_n' denotes dR_n/dx. By virtue of orthogonality, we get at once

$$(76)\qquad \int p(x)R_n\phi_i(x)dx = \begin{cases} 0, & (i \leqq n), \\ f_i, & (i > n). \end{cases}$$

This shows that

$$\int p(x)R_nG_n(x)dx = 0,$$

and the second relation in (75) yields the fundamental relation

$$(77)\qquad \int p(x)R_nG_1(x)R_n'dx = \int p(x)R_nG_1(x)f'(x)dx,$$

($G_1(x) = g_0 + g_1x$; g_i = arbitrary constants, independent of n).

In (77) integrate by parts on the left and apply Schwartz's inequality on the right. This gives

$$\left|\int p(x)R_nG_1(x)R_n'dx\right| = \left|\frac{1}{2}\left\{p(x)R_n^2G_1(x)\Big|_{x=a}^{x=b}\right.\right.$$

$$(78)\qquad \left.\left. - \int R_n^2\frac{p'(x)}{p(x)}p(x)G_1(x)dx - g_1\int p(x)R_n^2dx\right\}\right|$$

$$\leqq \int p(x)R_n^2dx \cdot \int p(x)f'^2(x)G_1^2(x)dx = o(1),$$

$$(n \to \infty;\ \text{by } (16)).$$

Moreover, by (73) and by use of the closure equation in its form (16),

$$\int p(x)R_n^2\cdot\frac{p'(x)}{p(x)}dx = o(1), \quad g_1\int pR_n^2dx = o(1), \quad (n \to \infty),$$

so that (78) leads to

$$p(x)R_n^2G_1(x)\Big|_{x=a}^{x=b} = o(1), \qquad (n \to \infty).$$

Taking here $G_1(x) = x - a$, $x - b$, we obtain

$$(79)\qquad p(b)R_n^2(b;f) = o(1), \quad p(a)R_n^2(a;f) = o(1), \quad (n \to \infty),$$

which, expressed in words, states that, *under conditions* (73) *and* (74) *the expansion* (6) *converges for* $x = a$, b *to the values* $f(a)$, $f(b)$, *respectively.**

* In some cases this alone is a sufficient basis for the conclusion that (6) converges *for all* x in (a, b).

In order to investigate R_n at any interior point, we need a further assumption as follows:

(80) $$f''(x) \text{ is of class } L_p^2 .$$

We then add to (75) the following relation:

$$f''(x) = \sum_{i=0}^{n} f_i\phi_i''(x) + R_n'', \qquad R_n'' = \frac{d^2}{dx^2} R_n,$$

and proceeding as above, we get

(81) $$\int p(x)R_nG_2(x)R_n''dx = \int p(x)R_nG_2(x)f''(x)dx = o(1), \qquad (n \to \infty).$$

Now take $G_2(x) = (x-a)(b-x)$, which is $\geqq 0$ in (a, b), and integrate by parts on the left side. Then

(82) $$\int \frac{p'(x)}{p(x)} p(x)R_nR_n'G_2(x)dx + \int p(x)R_nR_n'G_2'(x)dx + \int p(x)R_n'^2G_2(x)dx = o(1), \qquad (n \to \infty).$$

Here the middle term is $o(1)$, by (78), and

$$\left| \int \frac{p'(x)}{p(x)} p(x)R_nR_n'G_2(x)dx \right| \leqq h\gamma\left(\int \frac{p'(x)}{p(x)} R_n^2 dx \cdot \int p(x)R_n'^2 dx\right)^{1/2} = h\gamma \cdot o(1)(I_n)^{1/2},$$

$$h = \max\left| \frac{p'(x)}{p(x)} \right|, \qquad \gamma = \max |G_2(x)| \text{ in } (a, b),$$

$$I_n = \int p(x)R_n'^2G_2(x)dx,$$

so that (82) becomes

$$(I_n)^{1/2}\cdot o(1) + I_n = o(1), \qquad (n \to \infty),$$

whence the important result,

(83) $$I_n = \int p(x)R_n'^2(x-a)(b-x)dx = o(1), \qquad (n \to \infty).$$

Consider now the identity

$$p(x)\phi(x)R_n^2\Big|_a^x = \int_a^x [p(x)\phi(x)R_n^2]'dx = \int_a^x p'(x)\phi(x)R_n^2\,dx$$
$$+ \int_a^x p(x)\phi'(x)R_n^2\,dx + 2\int_a^x p(x)\phi(x)R_nR_n'\,dx$$
$$\equiv i_1 + i_2 + i_3, \qquad (\phi(x) = (x-a)(b-x)).$$

By the above considerations,

$$i_{1,2} = o(1) \text{ uniformly in } x, \quad (n\to\infty);$$
$$|i_3| \leqq 2\cdot\left(\frac{b-a}{2}\right)\left(\int p(x)R_n^2\,dx\cdot\int p(x)R_n'^2\phi(x)dx\right)^{1/2} = o(1),$$

whence we reach the final conclusion that

$$p(x)(x-a)(b-x)R_n^2(x) = o(1), \qquad (n\to\infty),$$

uniformly for $a\leqq x\leqq b$; that is, *under conditions* (73), (74) *and* (80), *the expansion* (6) *converges to* $f(x)$ *uniformly in the interval* $(a+\epsilon,\ b-\epsilon)$.

The method is applicable to a more general $p(x)$, of the form $(x-a)^{\alpha-1}(b-x)^{\beta-1}q(x)$, $(\alpha,\beta>0)$. We must point out, however, that the conditions imposed on $f(x)$ are far too stringent. This is compensated by the simplicity of the method, which requires neither the discussion of the coefficients f_n, nor the use of the asymptotic expression for $\phi_n(x)$. On the contrary, if we apply the above considerations to $f(x)\equiv\phi_{n+1}(x)$, we obtain directly

$$|\phi_n(c)| = (2n+1/p(c))^{1/2}(1+O(1/n)), \qquad (c=a,\ b).$$

3. *Use of the Differential Equations for the Classical OP.* These can be written as follows:

$$(84)\qquad A(x)\phi_n''(x) + B(x)\phi_n'(x) + C_n\phi_n(x) = 0,$$

where A and B are polynomials of degree $\leqq 2$, independent of n, and where C_n denote constants.

$$(85)\quad \begin{aligned} &(\text{J})\ (1-x^2)\phi_n''(x) + [\alpha-\beta-(\alpha+\beta)x]\phi_n'(x) \\ &\qquad\qquad + n(n+\alpha+\beta-1)\phi_n(x) = 0, \\ &(\text{L})\ x\phi_n''(x) + (\alpha-x)\phi_n'(x) + n\phi_n(x) = 0, \\ &(\text{H})\ \phi_n''(x) - 2x\phi_n'(x) + 2n\phi_n(x) = 0, \qquad (n=0,1,\cdots). \end{aligned}$$

From this we learn at once, by differentiation, that $\phi_n'(x)$ *is an OP of the same class, with* $p(x)$ *replaced by* $p_1(x)=A(x)p(x)$. Denoting the normalized $\phi_n'(x)$ by $\psi_{n-1}(x)$, we derive, by a simple computation, the following important relation which holds true for all classical OP:

$$\phi_n'(x) = (C_n)^{1/2}\psi_{n-1}(x), \qquad (n = 1, 2, \cdots), \tag{86}$$

where

$$\int p_1(x)\psi_m(x)\psi_n(x)dx = \delta_{mn}, \tag{87}$$

where $m, n=0, 1, \cdots$, and where $p_1(x)=A(x)p(x)$. Furthermore, it is readily seen that for all classical OP

$$(A(x)p(x))' = B(x)p(x), \qquad A(x)p(x) = 0 \text{ for } x = a, b. \tag{88}$$

Assuming now the existence in (a, b) of $f'(x)$ of the class L_p^2, (88) enables us to treat the coefficient f_n as follows. We write

$$\begin{aligned} C_nf_n &= \int p(x)f(x)C_n\phi_n(x)dx \\ &= \int p(x)f(x)[-A\phi_n''(x) - B\phi_n'(x)]dx, \end{aligned}$$

whence, by integration by parts, in virtue of (86) and (88),

$$f_n = \frac{f'_{n-1}}{(C_n)^{1/2}}, \tag{89}$$

where

$$f'_{n-1} = \int p_1(x)f'(x)\psi_{n-1}(x)dx, \text{ that is, } f'(x) \sim \sum_{n=0}^{\infty} f_n'\psi_n(x).$$

We have now

$$|f_n\phi_n(x)| = \left| f'_{n-1}\cdot\frac{\phi_n(x)}{(C_n)^{1/2}} \right| \leqq \frac{1}{2}\left\{ f'^2_{n-1} + \frac{\phi_n^2(x)}{C_n} \right\}.$$

Here the series $\sum_{n=1}^{\infty} f'^2_{n-1}$ certainly converges (closure), and

$$
(90)\quad
\begin{aligned}
&(\mathrm{J})\quad \phi_n^2(x)/C_n = O(n^{-2}), && (-1+\epsilon \leqq x \leqq 1-\epsilon);\\
&(\mathrm{L})\quad \phi_n^2(x)/C_n = O(n^{-3/2}), && \\
& && (0<\epsilon \leqq x \leqq A;\ A \text{ finite, fixed arbitrarily});\\
&(\mathrm{H})\quad \phi_n^2(x)/C_n = O(n^{-3/2}), && (|x| \leqq A).
\end{aligned}
$$

Hence, *the expansion* (6) *in series of classical OP converges uniformly and absolutely in the respective intervals as given in* (90), *if* $f'(x)$ *is of the class* L_p^2 *in* (a, b).

The same results could be obtained, without making use of the asymptotic expression for $\phi_n(x)$ (necessary to establish (90)), if we combine the use of the differential equation with that of closure as in the previous section (this would eliminate the introduction of $f''(x)$).

4. *Comparison Method.* The problem before us is: Given $p_1(x) \leqq p_2(x)$ in (a, b), where $p_1(x)$ and $p_2(x)$ are two weight-functions, find a corresponding qualitative relation between the partial sums of the two corresponding expansions (6) of the same function $f(x)$. This problem was solved by Szegö [28]. The basis of his analysis is the solution of the following *two-parameter* extremal problem.

Given in $(-1, 1)$ *a weight-function* $p(x)$ *and a function* $F(x)$ *such that* $F(x)/p(x)$ *and* $F^2(x)/p(x)$ *are both of class* L, *find* max $\{\lambda G_n(\xi)+\mu\int_{-1}^{1}F(x)G_n(x)dx\}^2$, ($\lambda,\mu$ *parameters*, ξ *a fixed point inside* $(-1,1)$), *for all* $G_n(x)$ *satisfying the condition* $\int_{-1}^{1}p(x)G_n^2(x)dx=1$.

Introduce the OP corresponding in $(-1, 1)$ to $p(x)$ and write

$$
(91)\quad
\begin{aligned}
G_n(x) &= \sum_{i=0}^{n} \gamma_i\phi_i(x),\ \gamma_i \text{ being constants, with } \sum_{i=0}^{n}\gamma_i^2 = 1.\\
\frac{F(x)}{p} &\sim \sum_{i=0}^{\infty} \alpha_i\phi_i(x), \qquad \alpha_i = \int_{-1}^{1} F(x)\phi_i(x)dx,\\
\frac{F(x)}{p(x)} &= \sum_{i=0}^{n} \alpha_i\phi_i(x) + R_n \equiv S_n\left(x;\ \frac{F}{p}\right) + R_n,\\
U_n(\lambda, \mu) &\equiv \left\{\lambda G_n(\xi) + \mu\int_{-1}^{1} F(x)G_n(x)dx\right\}^2\\
&= \left\{\sum_{i=0}^{n} \gamma_i[\lambda\phi_i(\xi) + \mu\alpha_i]\right\}^2, \qquad \sum_{i=0}^{n}\gamma_i^2 = 1.
\end{aligned}
$$

Then, by Cauchy's inequality and (13),

$$M_n(\lambda, \mu; p) \equiv \max U_n(\lambda, \mu) = \sum_{i=0}^{n} (\lambda\phi_i(\xi) + \mu\alpha_i)^2,$$

$$M_n(\lambda, \mu; p) = \lambda^2 K_n(\xi) + 2\lambda\mu \sum_{i=0}^{n} \alpha_i\phi_i(\xi) + \mu^2 \sum_{i=0}^{n} \alpha_i^2$$

$$(92) \qquad = \lambda^2 K_n(\xi) + 2\lambda\mu S_n\left(\xi; \frac{F}{p}\right) + \mu^2 U_n\left(\frac{F}{p}\right),$$

$$U_n\left(\frac{F}{p}\right) \equiv \sum_{i=0}^{n} \alpha_i^2 \leqq \int \frac{F^2(x)}{p(x)} dx \equiv U(p).$$

On the basis of the condition $\int_{-1}^{1} p(x) G_n^2(x) dx = 1$, we readily verify that

$$(93) \qquad \begin{gathered} p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1) \text{ implies} \\ M_n(\lambda, \mu; p_2) \leqq M_n(\lambda, \mu; p) \leqq M_n(\lambda, \mu; p_1). \end{gathered}$$

Hence, the difference $M_n(\lambda, \mu; p) - M_n(\lambda, \mu; p_2)$, which is a quadratic form in λ, μ, is positive for all real λ, μ. Expressing the fact that its discriminant is non-positive, we get, replacing $F(x)/p(x)$ by $f(x)$, of class L_p^2, and writing $S_n(x; p), K_n(x; p), U_n(p), \cdots$,

$$(94) \qquad \begin{aligned} &p(x) \leqq p_2(x) \text{ in } (-1, 1) \text{ implies} \\ &\quad [S_n(\xi; p) - S_n(\xi; p_2)]^2 \leqq U_n(p) \cdot [K_n(\xi; p) - K_n(\xi; p_2)] \\ &\qquad\qquad \leqq U(p)[K_n(\xi; p) - K_n(\xi; p_2)], \end{aligned}$$

where

$$U(p) = \int p(x) f^2(x) dx.$$

This is the desired relation between $S_n(x; p)$ and $S_n(x; p_2)$. Incidentally we have found here before (94) two other important relations of the same kind:

$$(95) \qquad p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1)$$

implies

$$K_n(\xi; p_1) \leqq K_n(\xi; p) \leqq K_n(\xi; p_2), \quad U_n(p_1) \leqq U_n(p) \leqq U_n(p_2).$$

It now follows that

$$(96)\qquad p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1)$$

implies

$$(S_n(\xi; p_1) - S_n(\xi; p_2))^2 \leqq U(p)[K_n(\xi; p_1) - K_n(\xi; p_2)].$$

Thus, if we succeed in choosing $p_1(x)$ and $p_2(x)$ in such a manner that

$$(97)\qquad \begin{aligned} &p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1), \\ &K_n(\xi; p_1) - K_n(\xi; p_2) = o(1), \ (n \to \infty;\ \xi \text{ given inside } (-1, 1)), \end{aligned}$$

then (94) *leads directly to an equiconvergence theorem*, namely,

$$(98)\qquad \lim_{n\to\infty} [S_n(\xi; p) - S_n(\xi; p_2)] = 0.$$

Szegö shows that such a choice of $p_1(x)$ and $p_2(x)$ is possible, if

$$(99)\qquad \begin{cases} p(x) = (1 + x)^{\alpha-1}(1 - x)^{\beta-1}q(x), \text{ with } 1/2 \leqq \alpha, \beta \leqq 3/2, \\ q(x) \text{ bounded and positive, } p(x) \text{ is } R\text{-integrable in } (-1, 1), \end{cases}$$

provided *at the interior point ξ under consideration $p(x)$ is continuous with its first and second derivatives.* We then can take

$$(100)\qquad p_1(x) = \frac{(1 - x^2)^{1/2}}{P_1(x)}, \quad p_2(x) = \frac{1}{(1 - x^2)^{1/2}P_2(x)},$$

where $P_1(x)$ and $P_2(x)$ are polynomials greater than zero in $(-1, 1)$; and we can choose the polynomials $P_1(x)$ and $P_2(x)$ so that, in addition to the first condition (97),

$$(101)\qquad 0 \leqq \int_{-1}^{1} \frac{\log p_2(x) - \log p(x)}{(x - \xi)^2} \cdot \frac{dx}{(1 - x^2)^{1/2}} < \epsilon; \qquad (p_1(\xi) = p(\xi) = p_2(\xi)).$$

Szegö then obtains an explicit expression for $\phi_n(x; p_{1,2})$(from a certain n on) using Fejér's theorem on the trigonometric representation of positive polynomials [29], and by means of Darboux's formulas shows that the second condition (97) is also satisfied. Incidentally we find in this way an asymptotic expression for $K_n(x; p)$ and $\phi_n(x; p)$. Thus, (98) is established.

But this is an intermediate step only. The desired equiconvergence theorem in its final form is

$$S_n(\xi; p) - \frac{\sigma_n(\xi)}{p(\xi)(1 - \xi^2)^{1/2}} \equiv o(1), \qquad (n \to \infty), \tag{102}$$

where $\sigma_n(\xi)$ represent the nth partial sum at the interior point $\xi = \cos\theta_0$ of the Fourier cosine-series expansion of the function $p(\cos\theta)f(\cos\theta)|\sin\theta|$. This is readily established, by first showing that (102) holds true for $p_2(x)$ (where the explicit expression of $\phi_n(x; p_2)$ is known) and then again using (96). Taking $q(x) \equiv 1$, we obtain the equiconvergence theorem for Jacobi polynomials, with $1/2 \leqq \alpha, \beta \leqq 3/2$.

IV. Illustration of the General Methods Discussed Above by Means of Legendre Polynomials.

The Legendre polynomials

$$\phi_n(x; 1) = \left(\frac{2n + 1}{2}\right)^{1/2} P_n(x),$$

where

$$\frac{1}{(1 - 2tx + t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

have, among others, the following properties:

$$\phi_n(-x) \equiv (-1)^n \phi_n(x); \ |\phi_n(x)| \leqq |\phi_n(+1)| = \left(\frac{2n + 1}{2}\right)^{1/2},$$

$(-1 \leqq x \leqq 1)$. Applying the preceding general methods to the expansion (6) of a given function $f(x)$ according to Legendre polynomials, we obtain the following results regarding its convergence.

(i) *Case of continuous functions.* From the estimate of $f_n\phi_n(x)$ we learn that (6) converges absolutely and uniformly in $(-1, 1)$ if $f'(x)$ satisfies therein a Lipschitz condition of order $> 1/2$. From the estimate of $R_n(x; f)$ we learn that (6) converges uniformly in any fixed interval $(-1+\epsilon, 1-\epsilon)$, if $f(x)$ satisfies in $(-1, 1)$ a Dini-Lipschitz condition, and uniformly in the whole interval $(-1, 1)$, if $f'(x)$ is therein continuous.

(ii) *Use of integral equations.* The kernel here is [10]

$$K(x, y) = \begin{cases} \log(1-y) + \log(1+x) + 1 - 2\log 2, \\ \qquad\qquad\qquad (-1 \leqq y \leqq x \leqq 1), \\ \log(1+x) + \log(1-y) + 1 - 2\log 2, \\ \qquad\qquad\qquad (-1 \leqq x \leqq y \leqq 1). \end{cases}$$

The series (6) converges absolutely and uniformly in $(-1, 1)$ if $f(x)$, $f'(x)$, and $f''(x)$ are therein continuous.

(iii) *Application of the Rademacher-Menchoff theorem.* The series (6) converges almost everywhere in $(-1, 1)$ if $f(x)$ is of bounded variation in $(-1, 1)$, or if

$$|f(x') - f(x'')| \cdot |\log|x' - x''||^{1+\epsilon} < \text{const.}, \qquad (-1 \leqq x', x'' \leqq 1).$$

(iv) *Use of Lebesgue constants.* Here

$$\rho_n(x) \sim \log n, \ (-1 < x < 1), \quad \rho_n(\pm 1) \sim n^{1/2}.$$

The series (6) converges uniformly in $(-1+\epsilon, 1-\epsilon)$ or in $(-1, 1)$, if $f(x)$ satisfies in $(-1, 1)$ a Dini-Lipschitz condition or a Lipschitz condition of order $>1/2$, respectively.

(v) *Use of the asymptotic expression of* $\phi_n(x)$. At any interior point $x = \cos\theta$, (6) behaves, regarding convergence or divergence, (a) like the ordinary Fourier series expansion of $f(\cos\theta)$, if $f(x)$ is of the class L^2 in $(-1, 1)$; (b) like the ordinary Fourier series expansion of $f(\cos\theta)|\sin\theta|$, if $f(x)$ is of the class L in $(-1, 1)$ and $\int_{-1}^{1}(1-x^2)^{-1/4}|f(x)|\,dx$ exists.

We may add that if we wish to discuss the effect of singularities of $f(x)$ at the end points $x = \pm 1$ on the convergence (or summability) properties of (6), the zero-series (p. 65) proves the most effective tool.

We see that the weakest method is that of integral equations, the strongest one, that of asymptotic expressions. Note that the latter enter explicitly or implicitly in many other methods dealing with the convergence of (6). The above considerations apply not only to ordinary convergence, but also to summability of (6), as was pointed out above. They also remain valid in many parts if we replace $p(x)dx$ by the more general $d\psi(x)$, where $\psi(x)$ is monotonic non-decreasing in (a, b), and use Stieltjes integrals.

We have had to omit in our discussion many other interesting topics. The most important are those that follow.

(i) *Gibbs' phenomenon*, familiar from the theory of ordinary Fourier series.

(ii) *The problem of Cantor.* Can we have $\sum_{n=0}^{\infty} a_n \phi_n(x) = 0$, $(a \leqq x \leqq b)$, without having $a_n = 0$ for all n? This is evidently equivalent to the problem of *uniqueness* of the expansion of a given $f(x)$ in series of orthogonal polynomials: can we have $\sum_{n=0}^{\infty} a_n \phi_n(x) = f(x)$, $\sum_{n=0}^{\infty} b_n \phi_n(x) = f(x)$ for certain x, without having $a_n = b_n$ for all n?

(iii) *The problem of Dubois-Reymond.* Assume the existence of $\int p(x) f(x) \phi_n(x) dx \equiv f_n$, $(n \geqq 0)$. If $\sum_{n=0}^{\infty} a_n \phi_n(x) = f(x)$ on a specified set $E \subset (a, b)$, can we conclude that $a_n = f_n$ for all n? This is equivalent to the problem *of term by term integration of the above series.*

These problems and many others on the general *OP* offer a vast and fruitful field of research. In some special cases (polynomials of Legendre, symmetric Jacobi polynomials), interesting results have been obtained by Plancherel and Kogbetliantz.

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