TERNARY ARITHMETICAL IDENTITIES*

BY E. T. BELL

1. *Introduction.* Two sets of identities are said to be equivalent when each set implies the other. By the method of paraphrase,† an elliptic theta identity is equivalent to one or more identities in functions with integer variables, the functions usually being subject to restrictions of parity (evenness or oddness in sets of variables). On account of their importance for class number relations‡ and other questions concerning quadratic forms in two or more variables, it is of interest to have the complete set of arithmetical identities equivalent to all the identities implied by the addition theorems for the thetas and the transformations of the first and second orders that are bilinear in thetas and doubly periodic functions of the second kind. Changes of q into $-q$, or increase of the variables by integer multiples of half periods in one of the latter identities, produce arithmetical equivalents obtainable immediately from the arithmetical equivalent of the original identity by proper specialization of the parity functions. Hence it is unnecessary to discuss such derived identities. Omitting these, we find precisely four identities to be paraphrased:

$$\sum \vartheta_1(-u+v+w)\phi_{100}(2w,-2v,q^2)=0; \tag{1}$$

$$\sum \vartheta_0(-u+v+w)\phi_{111}(2w,-2v,q^2)=0; \tag{2}$$

$$\sum \vartheta_1(-u+v+w)\phi_{122}(w,-v,q^{1/2})=0; \tag{3}$$

$$\sum \vartheta_3(-u+v+w)\phi_{111}(w,-v,q^{1/2})=0; \tag{4}$$

where the sum refers to the three products obtained in each case from the one written by the substitutions 1, uvw, uwv; and

$$\phi_{rst}(x,y)\equiv\phi_{rst}(x,y,q)\equiv\vartheta_1'\vartheta_r(x+y)/[\vartheta_s(x)\vartheta_t(y)],$$

the parameter in the thetas being q. The arithmetical equiva-

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† E. T. Bell, Transactions of this Society, vol. 22 (1921), pp. 1–30, 198–219. Cited as B.

‡ J. V. Uspensky, Bulletin de l'Académie des Sciences de Russie, 1925, pp. 599–620, 763–784; 1926, pp. 25–38, 175–196, 327–348. Cited as U.

lents of (1)–(4) are stated in §2, and several equivalents of these are indicated in §3.

To prove (1)–(4), we take

$$A = \vartheta_a(x), \qquad B = \vartheta_a(y), \qquad C = \vartheta_a(z),$$
$$X = \vartheta_b(x), \qquad Y = \vartheta_b(y), \qquad Z = \vartheta_b(z), \qquad a \neq b,$$

in the identity

$$\begin{vmatrix} A & B & C \\ X & Y & Z \\ X & Y & Z \end{vmatrix} = 0;$$

reduce the results, for $(a, b) = (1, 2), (0, 3), (1, 0), (3, 2)$, by the addition theorems for the thetas; introduce the doubly periodic functions of the second kind by dividing throughout by the appropriate product of the thetas; and finally make the change of variables

$$x = -u+v+w, \qquad y = u-v+w, \qquad z = w+v-w.$$

The addition theorems used are

$$\vartheta_1(x)\vartheta_2(y) - \vartheta_2(x)\vartheta_1(y) = 2\vartheta_0(x+y, q^2)\vartheta_1(x-y, q^2),$$
$$\vartheta_0(x)\vartheta_3(y) - \vartheta_3(x)\vartheta_0(y) = 2\vartheta_1(x+y, q^2)\vartheta_1(x-y, q^2),$$
$$\vartheta_1(x)\vartheta_0(y) - \vartheta_0(x)\vartheta_1(y) = \vartheta_2\left(\frac{x+y}{2}, q^{1/2}\right)\vartheta_1\left(\frac{x-y}{2}, q^{1/2}\right),$$
$$\vartheta_3(x)\vartheta_2(y) - \vartheta_2(x)\vartheta_3(y) = \vartheta_1\left(\frac{x+y}{2}, q^{1/2}\right)\vartheta_1\left(\frac{x-y}{2}, q^{1/2}\right).$$

The arithmetical equivalents of (1)–(4) are summarized at the end of §3.

2. *Arithmetical Functions.* An arithmetical function is one which is finite and single-valued for all sets of integer values of the variables. Let $f(x, y, z), \cdots, L(x, y, z)$ be arithmetical functions of x, y, z subject only to the following restrictions, in which x, y, z denote integers.

(5) $f(x, y, z) = f(-x, -y, -z).$

(6) $g(x, y, z) = -g(-x, -y, -z), \qquad g(0, 0, 0) = 0.$

(7) $F(x, y, z) = F(y, z, x) = F(z, x, y) = F(-x, -y, -z).$

(8) $G(x, y, z) = G(y, z, x) = G(z, x, y) = -G(-x, -y, -z),$

$$G(0, 0, 0) = 0.$$

(9) $H(x, y, z) = H(y, z, x) = H(z, x, y).$

(10) $K(x, y, z) = K(y, z, x) = K(z, x, y) = K(x, z, y).$

(11) $L(x, y, z) = L(y, z, x) = L(z, x, y) = -L(x, z, y).$

Thus $F(x, y, z)$ is any even arithmetical function of x, y, z which belongs to the cyclic group on x, y, z; $G(x, y, z)$ is any odd arithmetical function of x, y, z belonging to the same group; $H(x, y, z)$ is any arithmetical function of x, y, z belonging to this group; $K(x, y, z)$ is any arithmetical function of x, y, z belonging to the symmetric group on x, y, z; and $L(x, y, z)$ is any alternating arithmetical function of x, y, z. Note that these functions need not be defined when x, y, z are not all integers.

We can now state arithmetical equivalents of (1)–(4). The letters x, y, z denote variable integers, m, n constant integers; $(-1 \mid x) \equiv (-1)^{(x-1)/2}$; $\epsilon(a) = 1$, or 0, according as a is or is not, the square of an integer >0. The equivalents are numbered correspondingly to (1)–(4); thus (I_1) and (1) are equivalent, etc.

(I_1) $m = x^2 + 4yz$; $x \gtrless 0$, $y > 0$, $z > 0$; x, y, z odd:

$$\sum (-1 \mid x) F(-x, x-2z, x+2y) = 0.$$

(II_1) or (IV_1) $n = x^2 + yz$; $x \gtreqless 0$, $y > 0$, $z > 0$:

$$G(-x, x-z, x+y) = \epsilon(n)\Big[G(-n^{1/2}, n^{1/2}, 0) + \sum_{r=1}^{n^{1/2}-1}\Big\{G(-n^{1/2}, n^{1/2}, r) - G(-n^{1/2}, r, n^{1/2})\Big\}\Big].$$

(II_2) $m = x^2 + 4yz$; $x \gtrless 0$, odd; $y > 0$, $z > 0$:

$$G(-x, x-2z, x+2y) = \epsilon(m) \sum_{r=1}^{(m^{1/2}-1)/2} \big[G(-m^{1/2}, m^{1/2}, 2r-1) - G(-m^{1/2}, 2r-1, m^{1/2})\big].$$

(III_1) $m = x^2 + 4yz$; $x \gtrless 0$, odd; $y > 0$, $z > 0$:

$$-\sum (-1 \mid x)(-1)^{y+z} F(-x, x-2z, x+2y) = \epsilon(m) \sum_{r=1}^{(m^{1/2}-1)/2} (-1)^r \big[F(-m^{1/2}, m^{1/2}, 2r-1) - F(-m^{1/2}, 2r-1, m^{1/2})\big].$$

Thus (II_1) is equivalent to (4); (II_1) and (II_2) are together equivalent to (2); and it is evident that (I_1) is the special case $m\equiv 5 \bmod 8$ of (III_1). If $m\equiv 5 \bmod 8$ in (II_2), we get

$$(\mathrm{II}_3)\quad m = x^2+4yz;\, x \gtrless 0,\ \text{odd};\ y > 0,\ z > 0;\ m \equiv 5 \bmod 8:$$
$$\sum G(-x,\ x-2z,\ x+2y) = 0.$$

It will be seen in §4 that (II_1) is equivalent to the fundamental identity of Uspensky (U, loc. cit.), which was shown in a previous paper* to be equivalent to (2) written with different arguments (an unsymmetrical form.)

To indicate the proofs we transform (I_1), (III_1) into their equivalents in terms of f defined in (5), and (II_1), (II_2) into their equivalents in terms of g defined in (6).

The summations in what follows extend over some set, which need not be specified, of triples of integers x, y, z. The constants c_{xyz} do not involve the functions $f, \cdots, L$ occurring in a particular sum.

We see first that

$$(12)\qquad \sum c_{xyz}[f(x,y,z) + f(y,z,x) + f(z,x,y)] = 0,$$
$$(13)\qquad \sum c_{xyz}F(x,y,z) = 0$$

are equivalent. For (5) is the only restriction upon f and, by (7), $F(x, y, z) = F(-x, -y, -z)$. Hence f may be replaced by F in (12). Reduction of the result by (7) gives (13). Thus (12) implies (13). Conversely, $f(x, y, z)+f(y, z, x)+f(z, x, y)$, considered as a function, say $F'(x, y, z)$, of x, y, z, satisfies all the conditions (7) on $F(x, y, z)$. Hence (13) implies (12). Thus (12), (13) are equivalent. In the same way

$$(14)\qquad \sum c_{xyz}[g(x,y,z) + g(y,z,x) + g(z,x,y)] = 0,$$
$$(15)\qquad \sum c_{xyz}G(x,y,z) = 0$$

are seen to be equivalent.

Restating (I_1), (III_1) by means of (12), (13) as their f-equivalents we get them in the forms into which (1), (3) paraphrase immediately by the method of the paper (B). As the details are all simple routine which has been exemplified many times in that paper and in (B_2), we shall omit them. Similarly for (II_1), (II_2) and (14), (15).

* E. T. Bell, this Bulletin, vol. 32 (1933), pp. 682–687. Cited as B_2.

3. *Arithmetical Equivalents.* By (7), (9) we may replace $F(x, y, z)$ by $H(x, y, z)+H(-x, -y, -z)$ in (13); and by (8), (9), $G(x, y, z)$ may be replaced by $H(x, y, z)-H(-x, -y, -z)$ in (15). Hence (13), (15) imply

$$(16) \qquad \sum c_{xyz}[H(x, y, z) + H(-x, -y, -z)] = 0,$$

$$(17) \qquad \sum c_{xyz}[H(x, y, z) - H(-x, -y, -z)] = 0,$$

respectively. Considered as a function, say $H'(x, y, z)$, of x, y, z, $F(x, y, z)+G(x, y, z)$ satisfies the conditions (9) on $H(x, y, z)$. Hence we may replace $H(x, y, z)$ by $F(x, y, z)+G(x, y, z)$ in (16), (17). By (7), (8) the results reduce to (13), (15). Thus (13), (16) are equivalent, and likewise for (15), (17).

We next see that

$$(18) \qquad \sum c_{xyz}[H(x, y, z) - H(x, z, y)] = 0,$$

$$(19) \qquad \sum c_{xyz}L(x, y, z) = 0$$

are equivalent. The implication of (18) by (19) follows from (11), (9). To see that (18) implies (19), we note that $L(x, y, z) +K(x, y, z)$ satisfies the conditions (9). Hence we may replace $H(x, y, z)$ by $L(x, y, z)+K(x, y, z)$ in (18). Reducing by (10), (11), we get (19).

Application of the foregoing equivalences to the identities in §2 gives their equivalents. Note that in the identities in §2 the sign of x may be changed in any of the functions, and y, z may be interchanged. In this way we find that

$$(\mathrm{I}_2) \qquad \sum(-1 \mid x)\ L(-x, x-2z, y+2z) = 0$$

is equivalent to (I_1), for the partition as in (I_1), and that (II_3) is equivalent to

$$(\mathrm{II}_4) \qquad \sum L(-x, x-2z, y+2z) = 0$$

for the same partition. Hence, separating x modulo 4, and combining (I_2), (II_4), we see that (I_1) and (II_3) together are equivalent to the pair

$$(20) \quad m = x^2 + 4yz;\ x \gtrless 0,\ y > 0,\ z > 0;\ x, y, z \text{ odd};\ x \equiv j \bmod 4:$$
$$\sum L(-x, x-2z, x+2y) = 0,$$

where j is either 1 or -1.

For (II_1) we have the equivalents (21) (II_5) below, in which the partition is as in (II_1).

$$
(21)\quad \begin{aligned}
&\sum [H(-x, x-z, x+y) - H(-x, x+z, x-y)] \\
&= \epsilon(n)\Bigg[H(-n^{1/2}, n^{1/2}, 0) - H(-n^{1/2}, 0, n^{1/2}) \\
&\quad + \sum_{r=1}^{n^{1/2}-1}\Big\{H(-n^{1/2}, n^{1/2}, r) - H(-n^{1/2}, r, n^{1/2}) \\
&\quad -H(n^{1/2}, -n^{1/2}, -r) + H(n^{1/2}, -r, -n^{1/2})\Big\}\Bigg];
\end{aligned}
$$

$$
(II_5)\quad \begin{aligned}
L(-x, x-z, x+y) &= \epsilon(n)\Bigg[L(-n^{1/2}, n^{1/2}, 0) \\
&\quad + \sum_{r=1}^{n^{1/2}-1}\Big\{L(-n^{1/2}, n^{1/2}, r) - L(n^{1/2}, -n^{1/2}, -r)\Big\}\Bigg].
\end{aligned}
$$

Omitting the H-equivalents of (II_2), (III_1), we pass to the L-equivalent; the partition is as in (II_2), (III_1):

$$
(II_6)\quad \begin{aligned}
&-\sum\{(-1)^{y+z}(-1 \mid x) \mp 1\}L(-x, x-2z, x+2y) \\
&= \epsilon(m) \sum_{r=1}^{(m^{1/2}-1)/2} [\{(-1)^r \pm 1\}L(-m^{1/2}, m^{1/2}, 2r-1) \\
&\quad + \{(-1)^r \mp 1\}L(m^{1/2}, -m^{1/2}, -2r+1)],
\end{aligned}
$$

the upper or the lower signs being taken throughout. This is equivalent to (II_2), (III_1).

The L-equivalents appear to be the most convenient. In summary for these, (1)–(4) are equivalent to (20), (II_5), (II_6) of this section, L being defined by (11). As will be indicated next, (II_5) is equivalent to Uspensky's identity.

4. *Conclusion.* Let $k(x, y, z)$ be an arithmetical function of x, y, z satisfying the conditions

$$
(22)\quad k(x, y, z) = -k(-x, y, z) = k(x, -y, -z), \quad k(0, y, z) = 0.
$$

Then k satisfies the conditions (6), and hence g may be replaced by k in the g-equivalent of (II_5) or (II_1), since the last pair are equivalent. Again, if x, y, z in (22) be replaced by any linear homogeneous functions of x, y, z with constant integer coefficients not all zero, the transformed $k(x, y, z)$ is an arithmetical function of x, y, z satisfying (22). Conversely, if the determinant of the transformation is ± 1, the new k-identity implies the old (the restriction that the determinant be ± 1, and not merely $\neq 0$, which would suffice for the existence of the inverse transformation, is necessary to ensure that the transformed functions shall be arithmetical as defined here). These conditions are satisfied by the transformation

$$\begin{aligned} x &\rightarrow a_1x + a_2y + a_3z, \\ y &\rightarrow b_1x + b_2y + b_3z, \\ z &\rightarrow c_1x + c_2y + c_3z, \end{aligned} \quad \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} = \begin{Vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{Vmatrix},$$

which reduces the g-equivalent of (II_1) immediately to Uspensky's form. Uspensky's proof was entirely elementary. Mordell* gave an elementary proof of an equivalent of Uspensky's identity, and Oppenheim† gave an elementary proof of an identity which I showed‡ to be equivalent to Uspensky's. It may be mentioned that the 48 arithmetical expansions of functions $\phi_{rst}(x, y)$ which are not doubly periodic of the second kind, and which will be published shortly in the American Journal, lead at once to similar identities concerning quadratic forms in seven variables, for example $x^2+tv+yz+wu$. The general case concerns forms in $4s+3$ variables; the identities of the present paper correspond to $s=0$.

California Institute of Technology

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* L. J. Mordell, Journal of the London Mathematical Society, vol. 4 (1929), pp. 291–296.

† A. Oppenheim, Quarterly Journal of Mathematics, (2), vol. 2 (1931), pp. 230–233.

‡ E. T. Bell, this Bulletin, vol. 38 (1932), pp. 263–268.