MECHANICAL INVARIANTS OF THE SWEEPING-OUT PROCESS*

BY C. H. DIX

In this paper we prove the following theorem.

THEOREM. *If a general bounded distribution of positive mass in a closed connected region R is swept out on a surface S entirely enclosing R in its interior, then the center of gravity and the principal axes are invariants for the sweeping-out transformation.*

Let the distribution be given by $\Phi(e)$, which means the mass associated with the point set e. Then the potential is

$$V(M) = \int_R \frac{1}{\overline{MP}}\, d\Phi(e_P).$$

The coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the center of gravity of the distribution are, respectively,

$$\frac{1}{\Phi(R)}\int_R x_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R y_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R z_P d\Phi(e_P).$$

We have the following lemma.

LEMMA. *If Φ is such that a density ρ exists and $\nabla^2 V = -4\pi\rho$ is satisfied everywhere, then the sweeping-out on a level surface Σ of V entirely including R leaves the center of gravity and the principal axes invariant.*

The surface Σ is formed by setting $V=\delta>0$. Let P_0 be the center of gravity of the distribution Φ. Let R_0 be the lower bound of the radii of all spheres containing R with center P_0. If M_0 is a large integer, P on Σ, and $\delta=\Phi(R)(M_0R_0)^{-1}$, then we shall have $R_0(M_0-1)\leqq\overline{PP}_0\leqq R_0(M_0+1)$. Hence Σ lies in the spherical shell whose center is the point P_0 and whose bounding radii are $R_0(M_0-1)$ and $R_0(M_0+1)$.

Let u be any function harmonic in a closed region v containing P_0 whose boundary is the level surface Σ. Then, applying Green's Theorem, we have

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$$\int_\Sigma u \frac{\partial V}{\partial n} d\sigma - \int_\Sigma V \frac{\partial u}{\partial n} d\sigma = \int_v (V\nabla^2 u - u\nabla^2 V) d\tau = 4\pi \int_v u\rho d\tau.$$

Since V is a constant on Σ and u is harmonic inside Σ,

$$\int_\Sigma V \frac{\partial u}{\partial n} d\sigma = V \int_\Sigma \frac{\partial u}{\partial n} d\sigma = 0.$$

Hence

$$\int_\Sigma u \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v u\rho d\tau.$$

Let $u = x$. Then

$$\int_\Sigma x \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v x\rho d\tau = \bar{x}\Phi(R),$$

with similar relations for $\bar{y}$, $\bar{z}$, $\overline{xy}$, $\overline{yx}$, and $\overline{zx}$. The expression $(\partial V/\partial n)/(4\pi)$ is of course the surface density of the swept-out mass on Σ. So the lemma is proved.

The extension to a general distribution is made by taking the iterated volume average of the potential until the corresponding mass distribution is sufficiently smooth to give rise to a potential satisfying Poisson's equation.

The treatment of these average functions has been carried out by G. C. Evans in a forthcoming paper.* They are used to prove the fundamental theorem of F. Riesz on the mass associated with a sub- or super-harmonic function. Now assuming the Riesz Theorem, let $\{\rho_i(P)\}$ be the sequence of positive densities corresponding to the super-harmonic functions $\{V(r_i, r_i, r_i, r_i; M)\}$ which are the fourth volume averages of V over spheres of center M and radius r_i. For each of these density distributions the conditions of the hypothesis in the lemma are satisfied.

For a small value of δ selected as in the lemma, $V_i(M) = V(r_i, r_i, r_i, r_i; M)$ is constant with respect to i on Σ, since the spherical volume average of a harmonic function gives the same harmonic function. Hence the same level surface Σ may be used at each stage in the sequence. At each stage

$$\int_\Sigma u \frac{1}{4\pi} \frac{\partial V_i}{\partial n} d\sigma = \int_v u\rho_i d\tau = \int_\Sigma u \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v u d\Phi_i(e_P),$$

* G. C. Evans, *On potentials of positive mass*, Transactions of this Society, vol. 37 (1935), No. 2.

where

$$\Phi_i(e) = \int_e \rho_i d\tau.$$

The limit of the sequence of integrals we may denote by $\bar{u}\Phi(R)$. Since u is bounded and continuous in v, and the Φ_i are of uniformly bounded variation on v, we may apply the Helly-Bray* theorem to obtain the result

$$\bar{u}\Phi(R) = \int_v u d\Phi(e_P).$$

Hence the lemma is true if Φ is a general bounded positive dis tribution.

To prove the theorem consider a level surface Σ of V enclosing both R and S in its interior. Consider the following sweeping-out transformations: T_{RS} = sweeping-out of R on S, $T_{S\Sigma}$ = sweeping-out of S on Σ, and $T_{R\Sigma}$ = sweeping-out of R on Σ. Now $T_{R\Sigma} \equiv T_{S\Sigma}T_{RS}$. Furthermore, $T_{R\Sigma}$ and $T_{S\Sigma}$ leave the center of gravity invariant. Thus we have *center of gravity of distribution on* R = *center of gravity of distribution on* Σ = *center of gravity of distribution on* S. A similar argument handles the principal axes.

If we have a closed surface S' bounding a region R' for which the Green's function can be constructed, this Green's function is the potential of the negative unit mass that has been swept out from the pole on S' and the positive unit point mass at the pole. Concerning the distribution of this swept-out mass we may observe the following property which is a corollary of our theorem: *the swept-out point mass has its center of gravity at the pole and its principal axes of inertia are arbitrary.*

That the distributions arising from the sweeping-out of a point mass are not the only ones with indeterminate principal axes is immediate. Take for example four equal point masses at the vertices of a regular tetrahedron (or take a homogeneous cube). In the cube the three moments of inertia about axes through the center normal to the faces are all equal. The momental ellipsoid is therefore a sphere. The momental ellipsoid for the tetrahedron has the same form relative to the four lines

* H. E. Bray, Annals of Mathematics, (2), vol. 20 (1918), pp. 177–186; see p. 180.

through the center of gravity and the vertices and so is a sphere.

The statement of our main theorem can be given in more general form but our statement is chosen on account of its intuitive simplicity. The set R we may take as merely closed and bounded; S may be the frontier of a bounded domain, D, which contains R. Then the conclusion remains the same as we have stated it in the simpler case.

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A DECOMPOSITION THEOREM FOR CLOSED SETS*

BY G. T. WHYBURN

Let P be any local† topological property of a closed set such that if K is any compact closed set lying in a metric space, then the set of all non-P-points of K is either vacuous or such that its closure is of dimension >0. The following are examples of such properties: (i) local connectivity, (ii) regularity (Menger-Urysohn sense), (iii) rationality, (iv) being of dimension $<n$, (v) belonging to no continuum of convergence, (vi) belonging to no continuum of condensation. In fact, it will be noted that in each of these cases, every non-P-point of a compact set K lies in a non-degenerate continuum of non-P-points of K. We proceed to prove the following theorem.

THEOREM. *If N denotes the set of all non-P-points of a compact closed set K in a metric space and if K is decomposed upper semi-continuously‡ into the components of $\overline{N}$ and the points of $K-\overline{N}$, then every point of the hyperspace H is a P-point of H.*

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† For the purposes of the present paper we shall understand by a local property of a set K a point property P such that if some neighborhood of a point x in K has property P at x, then K has property P at x; and conversely, if K has property P at x, then any neighborhood of x in K also has property P at x. A point x of K will be called a P-point or a non-P-point of K according as K does or does not have property P at x.

‡ For the notions relating to upper semi-continuous decompositions and for a proof that our particular decomposition is upper semi-continuous, the reader is referred to R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society, Colloquium Publications, 1932, Chapter 5.