TOPICS IN THE FUNCTIONAL CALCULUS†

BY L. M. GRAVES

PART I. THE THEORY OF FUNCTIONALS

In this lecture it is proposed to outline an abstract theory of functionals, with a development paralleling that of the theory of functions of real variables, and including also a chapter on analytic functionals. In Part II, some applications of the general theory to various sorts of equations are indicated.

From the abstract point of view, the functional calculus is a form of general analysis, and as such it was effectually initiated by Fréchet's thesis in 1906. Since then a large number of researches have been concerned with the topological properties of abstract sets, and with the properties of continuous or semi-continuous functionals.

The postulational basis for an abstract topological theory may take various forms. A general basis consists of a general, that is, unrestricted class $\mathfrak{X}$ of elements x, and an unrestricted function K on $\mathfrak{S}$ to $\mathfrak{S}$, where $\mathfrak{S}$ is the class of all subsets E of $\mathfrak{X}$. The function K is then a set-valued function of sets. The system $(\mathfrak{X}, K)$ constitutes a topological space. It has been shown by Chittenden [7, pp. 294–298]‡ that a related set-function H may always be defined such that the space $(\mathfrak{X}, H)$ has the three properties:

I. $H(D+E)=H(D)+H(E)$.

II. For every set E, $H(E)$ contains $H(H(E))$.

III. If E is finite, $H(E)$ is null.

Such a space $(\mathfrak{X}, H)$ is called an *accessible space* by Fréchet.

If the points of $H(E)$ are called the *points of accumulation* of the set E, then closed sets may be defined as usual. A point of a set E is interior to E in case it is not a point of accumulation of the complement of E. Open sets are those consisting only of interior points. The neighborhoods of a point x may be defined as those sets having x as an interior point. A set E is called *com-*

† An address delivered by invitation of the program committee at the Chicago meeting of this Society, April 19, 1935.

‡ References in brackets are to the bibliography at the end of the paper.

pact in case every infinite subset A of E has at least one point of accumulation, that is, $H(A)$ is not null.

Another basis for a topological theory consists of the system $(\mathfrak{X}, \mathfrak{S}_0)$, where $\mathfrak{S}_0$ is a subclass of the class $\mathfrak{S}$ of all subsets E of $\mathfrak{X}$. If the sets E_0 in the class $\mathfrak{S}_0$ are thought of as the *open* subsets of $\mathfrak{X}$, we are led to definitions of point of accumulation and of derived set $E' = K(E)$ which give us a system $(\mathfrak{X}, K)$. Sierpiński [**29**, pp. 1, 28] has given five postulates for the system $(\mathfrak{X}, \mathfrak{S}_0)$ which are equivalent to the properties I, II, and III for the corresponding system $(\mathfrak{X}, K)$. Hausdorff [**17**, p. 213] considered a system $(\mathfrak{X}, \mathfrak{S}_0)$ in which the sets E_0 in the class $\mathfrak{S}_0$ are related as *neighborhoods* to the points of $\mathfrak{X}$. Hausdorff's set of four postulates is more restrictive than the set given above.

Still another basis for a topological theory is the system (class (L) of Fréchet) $(\mathfrak{X}, \Sigma_c, L)$, where Σ_c is a class of sequences of points of $\mathfrak{X}$, called *convergent sequences*, and L is a function on Σ_c to $\mathfrak{X}$, assigning to each convergent sequence its *limit* [Fréchet, **10**, p. 164]. Fréchet assumed:

(a) If $x_n = x$ for every n, then $L\, x_n = x$.

(b) If $L\, x_n = x$, then every subsequence converges and has the same limit.

In such a system derived sets $E' = K(E)$ are readily defined, yielding a topological space $(\mathfrak{X}, K)$. But it is possible for a compact sequence of distinct points (x_n) to have only one point of accumulation x without being convergent. However, this inconvenience may always be removed by extending the range Σ_c of definition of the function L to a maximum without changing the topological character of the space, that is, without changing the function K [Urysohn, **30**]. If in addition derived sets are always closed, the space $(\mathfrak{X}, K)$ obtained from $(\mathfrak{X}, \Sigma_c, L)$ is an accessible space. If this additional condition does not hold, a related function $H(E)$ may be defined by the process of Chittenden so that the resulting space $(\mathfrak{X}, H)$ is accessible. But a point of accumulation of a set E in the space $(\mathfrak{X}, H)$ need not be the limit of any convergent sequence of points from E. For example, let $\mathfrak{X}$ be the space of all real-valued functions $x(t)$ defined on $0 \leqq t \leqq 1$, and define $L\, x_n = x$ to mean ordinary convergence for each value of t. Then the related accessible space $(\mathfrak{X}, H)$ defined by the process of Chittenden (which in this case is also a Hausdorff space) is such that the derived set $H(E)$ of

a given set E contains all functions obtainable from those of E by *repetitions* of the limiting process. Thus if E is the class of continuous functions, $H(E)$ consists of all the Baire functions.

The notion of limit of a sequence may be defined in any accessible space $(\mathfrak{X}, H)$ thus: $L\, x_n = x$ in case every neighborhood of x contains all the points of the sequence from a certain place on. In the example just mentioned, this definition coincides with the original definition of limit. However, in a general accessible space, a sequence may have more than one limit. In a Hausdorff space on the other hand, a sequence can have at most one limit. But it is still possible for a set E to have a point of accumulation which is limit of no sequence of points from E.

In discussing functions defined on abstract spaces it seems convenient to refer to a *numerically-valued* function defined on an abstract space as an *operation*, and to use the word *transformation* to refer to general functional relations between two or more such spaces.

A function or transformation $f(x)$ which transforms an accessible space $\mathfrak{X}$ into another accessible space $\mathfrak{Y}$ is defined to be *continuous* at a point x of $\mathfrak{X}$ in case for every open set B containing $f(x)$ there exists an open set A containing x such that the transform $f(A)$ is contained in B. It follows that a continuous function of a continuous function is continuous. Also a continuous transform of a connected set is connected. If the space $\mathfrak{X}$ is compact and an operation f is continuous on $\mathfrak{X}$, then f is bounded on $\mathfrak{X}$. If every operation f continuous on $\mathfrak{X}$ is bounded on $\mathfrak{X}$, then for each such f the set of its functional values is closed. As a corollary we may say that a real-valued operation f continuous on a closed compact set E has a maximum and a minimum on E. A Borel theorem may be stated for an accessible space as follows. Let the set E be compact and closed, and let $\mathfrak{F}$ be a denumerable family of open sets O covering E, in the sense that each point of E is interior to a set O of the family $\mathfrak{F}$. Then there exists a finite subfamily of $\mathfrak{F}$ which also covers E [see Hildebrandt, **20**, pp. 470, 471; Chittenden, **7**, p. 300].

A metric space $(\mathfrak{X}, \rho)$ consists of a set $\mathfrak{X}$ of elements x and a real-valued operation ρ defined on $\mathfrak{X}\mathfrak{X}$, with the properties:

(1) $\rho(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

(2) $\rho(x_1, x_3) \leqq \rho(x_1, x_2) + \rho(x_3, x_2)$.

It follows from these two properties that the operation ρ is

symmetric and non-negative. If we agree that $L\ x_n = x$ when $\lim \rho(x_n, x) = 0$, we see that a metric space is a class $(\mathfrak{L})$ of Fréchet. It is also a Hausdorff space. It is plain that the distance function may be altered without changing the topological character of the space. For example, we may obtain a new distance function ρ^* by setting $\rho^* = \rho/(1+\rho)$. A fundamental problem is that of determining the conditions under which a topological space $(\mathfrak{X}, K)$ is metrizable. That is, when does there exist a metric ρ defining the same points of accumulation for a set as the relation K? For certain types of topological spaces the problem has been solved by Alexandroff and Urysohn, and by Chittenden [6].

In case the Cauchy condition, $\lim_{m,n} \rho(x_m, x_n) = 0$, is sufficient to ensure the convergence of a sequence (x_n) to a point x of a metric space $(\mathfrak{X}, \rho)$, the space is called *complete*. The property of completeness may be lost by a change of metric, so that it is not a topological property. However, we may consider the topological property of "completeness for some metric ρ." Fréchet prefers to apply the word "complete" to any topological space having this latter property. Likewise, from the topological point of view, a space which is metrizable is called metric. In the sequel we shall be interested in many non-topological properties, so that we shall use the term "metric space" to refer to a space $(\mathfrak{X}, \rho)$ with a definite associated metric ρ, and the term "complete" will be used only with reference to that metric, and not in the Fréchet sense.†

In a metrizable space the Borel theorem holds without the restriction of denumerability of the original family $\mathfrak{F}$ covering the set. Moreover, a continuous transformation f of a closed compact set in a metric space into a metric space is always uniformly continuous.

We have thus far surveyed the elementary ideas and theorems of point-set theory and the fundamental properties of continuous functions as they appear in the foundation of the abstract functional calculus or general analysis. Much of the theory of abstract sets has been passed by in order to adhere to the plan of outlining the analogies of the abstract functional calculus

† Von Neumann has recently given an interesting discussion of the notion of completeness for linear spaces which may not be metrizable. See Transactions of this Society, vol. 37 (1935), pp. 1–20.

with the elementary theories of functions of real and complex variables. We shall turn next to the consideration of vector spaces, and then develop the differential calculus of abstract functions.

A system $(\mathfrak{X}=[x], \mathfrak{A}=[a], +, \cdot, \| \ \|)$ which has the following properties will be called a *normed linear space.*

1. $\mathfrak{A}$ is the real number system or the complex number system.
2. $+$ is an associative transformation on $\mathfrak{X}\mathfrak{X}$ to $\mathfrak{X}$.
3. $\cdot$ is a transformation on $\mathfrak{A}\mathfrak{X}$ to $\mathfrak{X}$, associative with multiplication of numbers in $\mathfrak{A}$.
4. $\cdot$ is distributive with respect to $+$ and with respect to addition of numbers in $\mathfrak{A}$.
5. $1 \cdot x = x$ for every x.
6. $\| \ \|$ is an operation on $\mathfrak{X}$ to $\mathfrak{A}$.
7. $\|x_1+x_2\| \leqq \|x_1\| + \|x_2\|$.
8. $\|a \cdot x\| = |a| \, \|x\|$.
9. There is a unique point Θ in $\mathfrak{X}$ such that $\|\Theta\| = 0$.

A normed linear space is obviously a metric space if we set $\rho(x_1, x_2) = \|x_1 - x_2\|$. In case such a space is also complete, it has been called by Fréchet "un espace (D) vectoriel complet" or a Banach space. Fréchet prefers to distinguish between the "points" of space and the "vectors" associated with ordered pairs of points [**8**; **10**, pp. 123–146, 201–204]. Thus an "affine topological space" consists of a system $(\mathfrak{X}, K)$ with an associated system of vectors $(\mathfrak{Q}=[\xi], +, \cdot, \| \ \|)$ having certain properties. For most purposes I can see no gain in this distinction, since by selecting arbitrarily an "origin" Θ in the space $\mathfrak{X}$ a correspondence may be set up between the points x and vectors ξ. Omitting the distinction simplifies the notation and shortens the list of postulates.

In this connection it should be noted that in certain interesting linear spaces $(\mathfrak{X}, K)$ which are also metrizable, no metric can be defined in terms of a norm satisfying Postulate 8 along with all the others [Fréchet, **8**, p. 50]. Two of these are the space $\mathfrak{M}$ of all measurable functions with limit defined in terms of convergence in measure or approximate convergence, and the space (E_ω) of Fréchet consisting of all infinite sequences of numbers, that is, points in infinitely many dimensions, with

limit defined in terms of convergence of each coordinate. For this reason Fréchet and Banach have considered vector spaces of a more general type than the normed linear spaces, and have derived a few properties of linear and polynomial transformations of such spaces [**9**; **2**, pp. 20–52]. However, this intermediate domain seems not to have aroused much interest.

A transformation $G(x)$ of a normed linear space $\mathfrak{X}$ into a normed linear space $\mathfrak{Y}$ is called *linear* in case

$$G(a_1x_1 + a_2x_2) = a_1G(x_1) + a_2G(x_2)$$

for every pair of points x_1, x_2 and every pair of numbers a_1, a_2. In case such a transformation is continuous, it is also bounded or modular, that is, there is a number M such that $\|G(x)\| \leqq M\|x\|$ for every x. The greatest lower bound of effective M's is called the *bound* or *modulus* or *norm* of G and may be denoted by $M(G)$.

An additional concept which has been found useful in connection with linear spaces is that of *weak convergence*, as distinguished from convergence in terms of the norm. Schauder has proposed the following list of postulates to characterize the concept [**27**, p. 664]. Let $\mathfrak{X}$ be a normed linear space, and let wL be a single-valued function defined on a class Σ_w of sequences of elements of $\mathfrak{X}$, and with functional values in $\mathfrak{X}$. The sequences of Σ_w are called *weakly convergent*. The properties assumed are:

(a) If $\lim \|x_n - x\| = 0$, then $wL\ x_n = x$.

(b) If $wL\ x_n = x$, $wL\ x_n' = x'$, $\lim a_n = a$, $\lim a_n' = a'$, then $wL\ (a_nx_n + a_n'x_n') = ax + a'x'$.

(c) If a sequence (x_n) is weakly convergent, the sequence of norms $(\|x_n\|)$ is bounded.

(d) If $wL\ x_n = x$ and $\|x_n\| \leqq M$, then $\|x\| \leqq M$.

(e) Every subsequence of a weakly convergent sequence has the same weak limit.

Such a system $(\mathfrak{X}, \Sigma_w, wL)$ is plainly a class $(\mathfrak{L})$ of Fréchet.

In a given space, various definitions of weak convergence having the specified properties are usually possible. For example, weak convergence may always be identified with convergence in the norm. In this case the concept would be sterile. An important property which may sometimes be obtained by a proper choice of the definition is: (C_w) every bounded set is

weakly compact, that is, every sequence which is bounded in the norm has a weakly convergent subsequence.

In case we define weak convergence as follows: $wL\ x_n = x$ in case $\lim f(x_n) = f(x)$ for every continuous linear operation f defined on $\mathfrak{X}$, it may be shown that the weak limit x of a sequence (x_n) belongs to the extension of the set $[x_n]$ to be linear and closed [Banach, **2**, p. 134]. Also every continuous linear transformation G is weakly continuous, that is, $wL\ G(x_n) = G(x)$ whenever $wL\ x_n = x$ [**2**, p. 143].

A functional transformation G of a set X_0 into part or all of a space $\mathfrak{Y}$ is commonly called *completely continuous* in case it is continuous and transforms every bounded subset E of X_0 into a compact set $G(E)$. Let us say that G is *totally continuous* in case it transforms every sequence which is weakly convergent according to the definition just given into a sequence convergent according to the norm. It may be shown by examples that neither of these properties implies the other. However, if every bounded subset E of X_0 is weakly compact, then a totally continuous transformation G of X_0 is also completely continuous. On the other hand a completely continuous *linear* transformation G is also totally continuous [Banach, **2**, p. 143].

It may not be out of place to remark here that the spaces of "modular" functions considered by E. H. Moore in his second general analysis theory are all normed linear spaces. Professor Moore defined a sequence of modular functions ξ_n to be convergent "in mode one" in case the norms are bounded, and the sequence $\xi_n(p)$ converges for each p. He showed that convergence in mode one is equivalent to weak convergence as defined above, and that every bounded set of modular functions is always weakly compact.

An important theorem on sets of functional operations is the following. Let the sequence of operations $g_n(x)$ defined on the complete normed linear space $\mathfrak{X}$ have the following properties:

(1) $|\, g_n(a_1x_1+a_2x_2)\,| \leqq |a_1|\,|g_n(x_1)| + |a_2|\,|g_n(x_2)|$.

(2) For every n, $|\, g_n(x)\,|$ is continuous in x.

(3) For every x, $g_n(x)$ is bounded with respect to n.

Then the operations $|\, g_n(x)\,|$ are continuous in x uniformly with respect to n. A more general form of this theorem was proved by Hildebrandt† [**19**]. An immediate consequence of the theorem is

† The general form of the theorem follows readily by an indirect proof

the proposition that the limit $f(x)$ of a sequence of linear continuous transformations $f_n(x)$ convergent for every point of the space $\mathfrak{X}$ is also continuous. Another consequence is the proposition that a multilinear transformation which is continuous in each of its arguments separately is continuous in all its arguments together.

There are several conceptions of differentiation which are useful in the functional calculus. Perhaps the first to be used was the very general notion of a variation, which gave to the calculus of variations its name. This notion has passed through several stages, and we shall notice only one modern form of it. Let f be a functional transformation defined on an open set X_0 in a normed linear space $\mathfrak{X}$, with functional values in another such space $\mathfrak{Y}$. Suppose that

(D_1) for every element δx in $\mathfrak{X}$ there exists the derivative

$$\frac{d}{d\alpha} f(x + \alpha\delta x)\Big|^{\alpha=0} \equiv \delta f(x, \delta x).$$

This function $\delta f(x, \delta x)$ is then called the *variation* of the function f at the point x.

Note that we have used here the notion of the derivative of a function $g(\alpha)$ of a numerical variable whose functional values lie in a normed linear space. The definition of such a derivative is exactly analogous to the one used for numerically-valued functions and it has similar properties. A like statement is true concerning the Riemann integral of such a function $g(\alpha)$ [see Graves, **13**], and we shall have several occasions to use such integrals. Various definitions have also been given for the corresponding generalization of the Lebesgue integral, but we shall not need them.

It may readily be shown that the variation $\delta f(x, \delta x)$ of a functional transformation $f(x)$ as defined by statement (D_1) always has the property that it is homogeneous of the first degree in δx. But it may fail to have many of the usual properties asso-

from the special case stated above. The special case in turn is an easy corollary of the theorem that a complete metric space is of the second category, and the following theorem: If the sequence of non-negative and lower semi-continuous operations $f_n(x)$ is bounded with respect to n for each point x of a set X_0 of the second category, then there is a sphere K in the space $\mathfrak{X}$ on which the operations $f_n(x)$ are bounded uniformly with respect to x and n [Banach, **2**, p. 19].

ciated with derivatives of functions of a single variable, as is shown by the fact that every function $f(x)$ homogeneous of the first degree has a variation at the origin, $\delta f(\Theta, \delta x) = f(\delta x)$. In order to obtain a theory of differentiation with more content, we may suppose that the variation $\delta f(x, \delta x)$ of a functional transformation $f(x)$ has one or more of the following additional properties:

(D_2) $\delta f(x, \delta x)$ is linear in δx.

(D_3) $\delta f(x, \delta x)$ is continuous in δx.

$$(D_4) \quad \lim_{\|\delta x\|=0} \frac{\|f(x+\delta x) - f(x) - \delta f(x, \delta x)\|}{\|\delta x\|} = 0.$$

(D_5') $\delta f(x, \delta x)$ is defined and continuous with respect to x at each point x in X_0 uniformly for $\|\delta x\| = 1$.

In case a property (D) holds for every point x of X_0 we shall denote the corresponding property by (D'). It may be shown that a function $\delta f(x, \delta x)$ having the relation to $f(x)$ expressed in properties (D_2) and (D_4) always has property (D_1) also. Moreover, properties (D_1'), (D_2'), and (D_5') imply (D_4'), provided the space $\mathfrak{Y}$ of functional values is complete. When (D_2), (D_3), and (D_4) are satisfied, we say that the function f has a *total differential* $\delta f(x, \delta x)$ at the point x. The importance of the notion of total differential has been emphasized by Fréchet on various occasions. For example, a function f having a total differential at a point x_0 is always continuous at x_0. Moreover, if also $g(z)$ has a total differential at z_0, and $g(z_0) = x_0$, then the function $f[g(z)]$ has a total differential at z_0, equal to $\delta f[x_0, \delta g(z_0, \delta z)]$. These theorems do not hold for functions having only variations in the weak sense expressed by condition (D_1) alone. However, we shall see later that when the function f is continuous and the derivative in (D_1) is taken with respect to a complex variable α, then property (D_1') implies all the others, provided again that the space $\mathfrak{Y}$ of functional values is complete.

The differentiability properties of a functional operation or transformation $f(x)$ may change when the continuity properties of the space $\mathfrak{X}$ are changed by altering the norm $\|x\|$. For an example we may take a calculus of variations integral

$$f[x] = \int_0^1 \phi(t, x, x')dt,$$

where the space $\mathfrak{X}$ consists of all functions $x(t)$ of class C' and vanishing at $t=0$ and $t=1$, and $\|x\| = \max |x(t)|$. Then

$$\delta f[x, \delta x] = \int_0^1 \{\phi_x \delta x + \phi_{x'} \delta x'\} dt$$

has properties (D_1) and (D_2). Property (D_3) also holds when $x(t)$ is of class C'', but an example shows that it may fail if $x(t)$ is merely of class C'. Property (D_4) fails in general, since the functional operation $f[x]$ is not usually even continuous. However, if we set $\|x\| = \max \{|x(t)|, |x'(t)|\}$, then in our example $\delta f[x, \delta x]$ has all of properties (D_2'), (D_3'), (D_4'), (D_5') in a suitably chosen region X_0, and the operation $f[x]$ is itself continuous.

The variety of useful definitions of differentiation is greater when we consider variations of higher order. There is time to mention only a few. We shall say that a transformation $f(x)$ has an nth variation $\delta^n f(x, \delta x)$ at a point x in case for every δx there exists the nth derivative

$$\frac{d^n}{d\alpha^n} f(x + \alpha \delta x)\Big|^{\alpha=0} \equiv \delta^n f(x, \delta x).$$

The nth variation $\delta^n f(x, \delta x)$ is always homogeneous of degree n in δx. Moreover, if we set

$$R_{n+1}(x, \delta x) = f(x + \delta x) - f(x) - \delta f(x; \delta x) - \cdots - \frac{1}{n!} \delta^n f(x; \delta x),$$

we have

$$\lim_{\alpha=0} \frac{\|R_{n+1}(x, \alpha \delta x)\|}{\alpha^n} = 0.$$

There is valid also a generalization of Taylor's theorem [Graves, **13**, p. 173], namely, if $f(x)$ is defined and has an nth variation at each point of a convex region X_0, and if the space $\mathfrak{Y}$ in which the functional values lie is complete, and if x and $x+\delta x$ lie in X_0, and $\delta^n f(x+t\delta x, \delta x)$ is a continuous function of t, then

$$R_n(x, \delta x) = \int_0^1 \delta^n f(x + t\delta x, \delta x)(1 - t)^{n-1} dt/(n - 1)!.$$

We might also say that a transformation $f(x)$ has a second variation at x_0 in case f has a first variation $\delta f(x, \delta x)$ for each x near x_0, and $\delta f(x, \delta x)$ has a first variation $\delta^2 f(x_0, \delta x, \delta_1 x)$ at x_0 for every δx. The nth variation would then be defined by induction, and would involve n independent variations $\delta_1 x, \cdots, \delta_n x$. In case $\delta f(x, \delta x)$ has the properties (D_2'), (D_3'), and (D_4') of a total differential, and $\delta^2 f(x_0, \delta x, \delta_1 x)$ has the properties (D_2), (D_3), and (D_4) for each δx, according to Fréchet $\delta^2 f(x_0, \delta x, \delta_1 x)$ is called the *second differential* of f at x_0. In this case it may be shown that $\delta^2 f(x_0, \delta x, \delta_1 x)$ is a symmetric function of δx, $\delta_1 x$, and hence that it is linear and continuous in δx and in $\delta_1 x$. When the space $\mathfrak{X}$ is complete, it follows that $\delta^2 f(x_0, \delta x, \delta_1 x)$ is continuous in δx, $\delta_1 x$ together.

A function $f(x_1, \cdots, x_k)$ of a finite number of real variables is commonly said to be of class $C^{(n)}$ on an open region in the x-space when it has continuous partial derivatives on that region up to and including those of the nth order. We may generalize this notion to our abstract situation as follows [Hildebrandt and Graves, **18**, pp. 135–144]. We say that $f(x)$ is of class C' on an open region X_0 in case f has a total differential $\delta f(x, \delta x)$ at every point of X_0, which also satisfies condition (D_5'). The function f is of class $C^{(n)}$ on X_0 if f is of class C' and $\delta f(x, \delta x)$ is of class $C^{(n-1)}$ uniformly for $\|\delta x\| = 1$. When $n \geqq 3$, this implies, for example, that $\delta^3 f(x, \delta_1 x, \delta_2 x, \delta_3 x)$ is continuous in $\delta_3 x$ uniformly for $\|\delta_1 x\| = \|\delta_2 x\| = 1$. Using this definition it is possible to prove, for example, that if $f(x)$ is of class $C^{(n)}$ on X_0, and if $g(z)$ is of class $C^{(n)}$ on Z_0 and has its functional values in X_0 then the function $h(z) = f[g(z)]$ is of class $C^{(n)}$ on Z_0 [Hildebrandt and Graves, **18**, p. 144]. While the proof of this theorem is rather difficult, the other formal rules of the differential calculus which have meaning in this general situation are readily derived.

In case $\mathfrak{X}$ and $\mathfrak{Y}$ are complete normed linear spaces with the complex number system $\mathfrak{A}$ as the associated number system, we may develop an interesting theory of analytic functional transformations f on a region X_0 to $\mathfrak{Y}$. The essential features of this were indicated in certain special cases by Gateaux [**11**, **12**]. Let us consider first the case of a function $f(z)$ on A_0 to $\mathfrak{Y}$, where A_0 is an open set in the complex plane. The definition of derivative of $f(z)$ and of ordinary point for $f(z)$ have the usual

form. A function $f(z)$ is said to be *holomorphic* on a set S in case every point of S is an ordinary point for $f(z)$. Line integrals of such functions have the usual elementary properties, and Cauchy's theorem and integral formula hold true. A Laurent expansion for $f(z)$ about an isolated singularity may be derived in the usual way, and the notion of analytic continuation also generalizes. All this was indicated by Wiener [31].

Let us turn now to the general case of a functional transformation $f(x)$ defined on an open set X_0 in a normed linear space. We say that a point x_0 is an ordinary point for f in case f is continuous and has a first variation at each point x near x_0. If each point of X_0 is an ordinary point for $f(x)$, then f is *holomorphic* on X_0.

A transformation $f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ is called a *polynomial transformation* in case $f(x_1+\lambda x_2)$ is a polynomial in the numerical variable λ with coefficients in the space $\mathfrak{Y}$, for every x_1 and x_2 in $\mathfrak{X}$. This definition reduces to the ordinary meaning of the word polynomial in case f is a numerically-valued function of a finite number of numerical variables. A continuous and homogeneous polynomial transformation $P_n(x)$ of degree n may be polarized to give a multilinear form $Q(x_1, x_2, \cdots, x_n)$ linear and continuous in each of its arguments, such that the relation $Q(x, \cdots, x) = P_n(x)$ holds. The kth differential of $P_n(x)$ is then $Q(\delta_1 x, \cdots, \delta_k x, x, \cdots, x)$ if $k \leqq n$, and vanishes if $k > n$.

Let the transformation $f(x)$ be defined by a series of the form

$$f(x) = \sum_0^\infty P_n(x). \tag{1}$$

There is a theorem on the nature of the region of convergence of such a series, analogous to Abel's theorem for ordinary power series. Let X_0 denote the open set remaining after discarding the boundary points of the set on which the series (1) converges. Then if the point x is in X_0 and $|\lambda| \leqq 1$, the point λx is also in X_0. We shall call a region X_0 having this property a *region of circular type* about the origin. In order to obtain interesting properties, we assume that the series (1) converges uniformly on sets E which are compact on X_0. An equivalent condition states that every point x_0 of the region X_0 has a neighborhood on which the series converges uniformly. It follows that the sum

$f(x)$ of the series is holomorphic on X_0, and in fact of class $C^{(\infty)}$.

Conversely, we have also a Taylor's theorem. That is, if a transformation $f(x)$ is holomorphic on X_0, then about each point x_0 in the region X_0 the transformation $f(x)$ is expansible in a series

$$f(x) = \sum_0^\infty P_n(x_0; x - x_0) \tag{2}$$

of continuous homogeneous polynomial functions of $(x-x_0)$, converging uniformly on a neighborhood of x_0, so that $f(x)$ is necessarily of class $C^{(\infty)}$. In case X_0 is a region of circular type about the point x_0, the Taylor expansion (2) for $f(x)$ converges uniformly on sets E compact on X_0. Also $\|f(x)\|$, being continuous, is bounded by a constant M for $\|x-x_0\| \leqq \rho$, where ρ is sufficiently small, so that by Cauchy's integral formula,

$$\|P_n(x_0; x - x_0)\| \leqq M(\|x - x_0\|/\rho)^n.$$

The sum of a series of functional transformations, $f(x) = \sum f_n(x)$, where each $f_n(x)$ is holomorphic on X_0, is also holomorphic on X_0, provided the series converges uniformly on sets E compact on X_0. Moreover, if f on X_0 to $\mathfrak{Y}$ is holomorphic on X_0 and g on U_0 to X_0 is holomorphic on U_0, then the transformation $h(u) = f[g(u)]$ is holomorphic on U_0. If $f(x, u)$ is a function of two points x and u which is continuous in x and u together and holomorphic with respect to x and u separately for x in X_0 and u in U_0, then $f(x, u)$ is a holomorphic function of the composite variable (x, u), and conversely. Finally, we may make use of the process of analytic continuation. However, we do not expect to secure much information about the nature of the singularities of an analytic functional transformation $f(x)$.

Part II. Implicit Function Theorems and Applications

We may apply the preceding theory to study the solutions of various sorts of equations. Thus it may be desired to solve the equations

$$f_i(x_1, \cdots, x_m, y_1, \cdots, y_n) = 0, \qquad (i = 1, \cdots, n),$$

for the y's as functions of the x's. This system of equations may be written $f(x, y) = 0$, where the range of x is a region of the m-dimensional x-space, the range of y is a region of the n-dimensional y-space, and the functional values of f are in the y-space. The differential equation $f(x, y, dy/dx) = 0$ has for the unknown a function $y = y(x)$ in the space of functions having continuous derivatives. The dependence of the solution $y(x)$ on the function f may be investigated by the methods of the functional calculus [Bliss, **4**]. From our general point of view, algebraic equations, differential equations, integral equations, and many other types are all gathered under one head. However, we may still wish to consider a variety of hypotheses, and there are different methods available.

The simplest method is that of successive approximations. Let $(\mathfrak{Y}, \rho)$ be a complete metric space, and let $(y_0)_a$ denote the neighborhood of y_0 consisting of those points y for which $\rho(y, y_0) < a$. Consider an equation of the form

$$y = F(y), \tag{3}$$

where the transformation F is a function on $(y_0)_a$ to $\mathfrak{Y}$ with the properties:

(a) It satisfies a Lipschitz condition with constant $k < 1$, that is, $\rho(F(y_1), F(y_2)) \leqq k\rho(y_1, y_2)$.

(b) $\rho(F(y_0), y_0) < (1-k)a$.

Then the method of successive approximations shows at once that there is a unique solution of equation (3) in the neighborhood $(y_0)_a$; that is, there is a unique *fixed point* of the transformation $\bar{y} = F(y)$. In case the transformation F depends on a parameter x, which may be a point in an accessible or a metric space, the solution of equation (3) becomes a function $y = Y(x)$, and the theorem is properly called an implicit function theorem. If the function $F(x, y)$ is continuous in its two arguments, then the solution $y = Y(x)$ will also be continuous [Hildebrandt and Graves, **18**, pp. 133–135].

Suppose we wish to consider an equation of the form $G(y) = \Theta$, where G is a transformation defined on a neighborhood $(y_0)_a$ with functional values in a space $\mathfrak{Z}$, and $\mathfrak{Y}$ and $\mathfrak{Z}$ are complete normed linear spaces. We may apply a generalization of Newton's method to obtain a solution (a process which is equivalent to reducing the equation to the form (3) and applying the

method of successive approximations), provided: (a) the point y_0 is a sufficiently close approximation to a solution of the equation, (b) the function G has a differential $\delta G(y_0, \delta y)$ at y_0, and (c) this linear transformation $\delta G(y_0, \delta y)$ on $\mathfrak{Y}$ to $\mathfrak{Z}$ has an inverse.

The question of the existence and properties of an inverse for a linear transformation G has been the subject of extensive investigations. I shall mention only a few elementary theorems. The first states that if a continuous linear transformation G has an inverse H, then H is also linear and continuous [Banach, **2**, p. 41]. The second states that if the space $\mathfrak{Y}$ is complete, and if the continuous linear transformation G of $\mathfrak{Y}$ into all or part of itself has norm $M(G)$ less than unity, then the transformation $K(y) = y - G(y)$ always has an inverse which may be expressed in terms of the usual series of iterates of G [Banach, **1**, p. 161; Hildebrandt and Graves, **18**, p. 145]. The third may be stated as follows. Let the space $\mathfrak{Y}$ be complete, and let $G(y, w)$ be a continuous linear transformation of $\mathfrak{Y}$ into $\mathfrak{Z}$ which depends also on a parameter w ranging over an accessible space. Suppose that G is continuous in w at $w = w_0$ uniformly for $\|y\| = 1$, and that $G(y, w_0)$ has an inverse. Then for w in a sufficiently restricted neighborhood of w_0, $G(y, w)$ has an inverse $H(z, w)$, and H is continuous at $w = w_0$ uniformly for $\|z\| = 1$ [Hildebrandt and Graves, **18**, p. 146].

A general implicit function theorem may be stated as follows. Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be normed linear spaces, of which $\mathfrak{Y}$ and $\mathfrak{Z}$ are also complete. Suppose that $G(x,y)$, with functional values in $\mathfrak{Z}$, is of class $C^{(n)}$ on a region R in the composite space $(\mathfrak{X}, \mathfrak{Y})$, and let (x_0, y_0) be an initial solution of the equation

$$G(x, y) = \Theta \tag{4}$$

at which the partial differential $\delta_y G(x_0, y_0; \delta y)$ has an inverse. Then in a sufficiently small neighborhood $(y_0)_a$ the equation (4) has a unique solution $y = Y(x)$ defined on a neighborhood $(x_0)_b$, and the solution $Y(x)$ is also of class $C^{(n)}$ [Hildebrandt and Graves, **18**, p. 150].

From our discussion of analytic functional transformations it is plain that when the transformation $G(x, y)$ is analytic, the solution $y = Y(x)$ is also analytic.

It is frequently of interest to know how far a solution $y = Y(x)$ of equation (4) may be extended. Let us define a *sheet of points*

in the composite space $(\mathfrak{X}, \mathfrak{Y})$ to be an arcwise connected set S with the additional property that for every point (x_1, y_1) of S there are positive constants a and b such that no two points of S in the neighborhood $(x_1, y_1)_a$ have the same projection x, and every point x in the neighborhood $(x_1)_b$ is the projection of a point (x, y) of S contained in $(x_1, y_1)_a$. Then under the hypotheses of the theorem just stated there is a uniquely determined maximal sheet S of solutions of equation (4) which (a) passes through the initial solution (x_0, y_0); (b) is composed entirely of points (x, y) at which the differential $\delta_y G(x, y; \delta y)$ has an inverse; and (c) has for its boundary points only boundary points of the region R and points at which $\delta_y G(x, y; \delta y)$ ceases to have an inverse. This theorem is readily proved by defining S as the logical sum of all sheets having the properties (a) and (b). It may be noted that a sheet of solutions determines a single-valued function $Y(x)$ in a neighborhood of each of its points, and each of these functions is of class $C^{(n)}$.

Various applications of the general implicit function theorem described above have been made to differential, integral, and integro-differential equations, and to the calculus of variations [Graves, **14**, pp. 535 ff.; **15**; **16**, p. 677]. The theorem is a very powerful one, and simple in application. The principal problem in any special case is to determine when the partial differential $\delta_y G(x, y; \delta y)$ has an inverse. Let us consider a simple application to an ordinary differential equation with boundary conditions,

$$(5) \qquad y'' = h(t, y, y'), \qquad y(a) = A, \qquad y(b) = B.$$

Let $\mathfrak{Y}$ be the space of all real functions $y(t)$ of class C'' on $a \leqq t \leqq b$, with $\|y\|$ defined in the usual way as the maximum of $|y(t)|, |y'(t)|, |y''(t)|$ on the interval. Let $\mathfrak{H}$ be the space of all real functions $h(t, y, y')$ continuous in t and of class $C^{(m)}$ in y and y', for (t, y, y') in a certain bounded region R. Let $\|h\|$ be the maximum of $|\partial^{i+j} h/\partial y^i \partial y'^j|$ for (t, y, y') in R and $i+j = 0, \cdots, m$. Let $\mathfrak{Z}$ be the composite space consisting of all continuous functions $z(t)$ and pairs of numbers A, B, with $\|z, A, B\|$ = maximum of $|z(t)|, |A|, |B|$. Then the equations

$$y'' - h(t, y, y') = z(t), \qquad y(a) = A, \qquad y(b) = B,$$

define a transformation $G(y, h)$ of class $C^{(m)}$ on a region of the composite space $(\mathfrak{Y}, \mathfrak{H})$ to the space $\mathfrak{Z}$. In this special problem

a necessary and sufficient condition that the partial differential $\delta_y G(y, h; \delta y)$ have an inverse transformation is that the equation of variation

$$(6) \qquad \delta y'' - h_{y'}\delta y' - h_y \delta y = 0, \qquad \delta y(a) = 0 = \delta y(b),$$

be incompatible, that is, have only the solution $\delta y \equiv 0$. Hence if an initial solution of equation (5) is known at which (6) is incompatible, this solution may be continued, allowing the function h and the end values A and B to vary until a point is reached which lies on the boundary of the region of admissible points, or at which equation (6) becomes compatible. This solution $y(t) = Y[h, A, B \mid t]$ will be of class $C^{(m)}$ as a function of h, A, and B.

This result may be applied to the calculus of variations as follows. Let $y = y(t)$ be an extremal for an integral

$$\int_a^b f(t, y, y')dt$$

joining two points (a, A) and (b, B). Suppose that

$$(7) \qquad f_y, \qquad f_{y't}, \qquad f_{y'y}, \qquad f_{y'y'}$$

have continuous partial derivatives up to order m with respect to y and y', and let $\|f\|$ be defined as the maximum of the absolute values of the partial derivatives (7) and their partial derivatives up to order m. Suppose that $f_{y'y'} \neq 0$ along the initial extremal, and that the end points of this extremal are not conjugate. Then when A, B, and the integrand f are varied, the extremal joining (a, A) and (b, B) will vary continuously and be of class $C^{(m)}$ as a function of f, A, and B, and the process may be continued until an extremal arc is reached which either is singular, that is, has $f_{y'y'} = 0$ at some point, or else has end points which are conjugate. If the initial extremal is a minimizing extremal, then the varied extremal will continue to furnish at least a weak relative minimum for the integral.

The method described may be applied also to partial differential equations of elliptic type. An essential theorem for that purpose is the one which states that if a linear elliptic partial differential equation of the second order has at most one solution taking given boundary values, then it always has a solution for arbitrary continuous boundary values, at least when the

shape of the boundary is sufficiently restricted. Schauder has recently given a proof of this based on the functional calculus and on the known solvability of the equation

$$(8) \qquad D(u) \equiv u_{xx} + u_{yy} = f(x, y), \qquad (u = \phi \text{ on the boundary}),$$

and known inequalities affecting the solutions [28]. Schauder gives the theorems for the case of n independent variables, but for simplicity they are stated here for the case $n=2$. Inequalities affecting the solutions of the equation

$$(9) \qquad L(u) \equiv a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y), \qquad (u = \phi \text{ on the boundary}),$$

where $a>0$, $ac-b^2=1$, are first derived. All the functions appearing in (9) are supposed to be α-Hölder continuous, $(0<\alpha<1)$, and the norms $\|u\|$, $\|f\|$, and $\|\phi\|$ involve the Hölder constants of those functions. The coefficients a, b, and c are supposed to be β-Hölder continuous, where $\beta>\alpha$. The inequalities proved yield a type of weak compactness of the set of solutions of the one-parameter family of equations

$$(10) \qquad K(\lambda, u) \equiv \frac{(1-\lambda)D(u) + \lambda L(u)}{P(\lambda)} = f, \qquad (u = \phi \text{ on the boundary}),$$

where $P(\lambda)$ is a certain function which is continuous and not zero for $0 \leqq \lambda \leqq 1$, and $\|f\|$ is bounded. Hence if (10) has a solution $u=u_n$ for $\lambda=\lambda_n$, and λ_n approaches λ, then a subsequence of the sequence u_n is weakly convergent to a solution u which also satisfies the inequalities. Now the transformation $K(\lambda, u)$ is continuous in λ and u and linear in u, so that it will continue to have an inverse for λ near a value λ_0 for which it is known to have an inverse. Since the family (10) reduces to equation (8) for $\lambda=0$ and to equation (9) for $\lambda=1$, and equation (8) always has a unique solution, it follows that equation (9) always has a unique solution. Let the inverse transformation so defined be denoted by $u=T(f)$. Then the general elliptic linear equation

$$(11) \qquad L(u) + d(x, y)u_x + e(x, y)u_y + g(x, y)u = h(x, y), \qquad (u = \phi \text{ on the boundary}),$$

may be transformed into

$$(12)\quad f = h - d\frac{\partial}{\partial x}T(f) - e\frac{\partial}{\partial y}T(f) - gT(f) \equiv h + S(f),$$

and the transformation $S(f)$ turns out to be completely continuous. Thus by use of the Riesz theory of linear transformations of the form (12), where $S(f)$ is completely continuous [Riesz, **24**; Hildebrandt, **21**; Schauder, **26**], it follows that if (11) has at most one solution, then it always has a solution.

The foregoing proofs are made first for the case when the boundary values ϕ are supposed to have continuous second derivatives satisfying a Hölder condition, and the theorems are extended to the case of arbitrary continuous boundary values by use of certain additional inequalities. This proof by Schauder is a good example of the advantages offered by the general methods of the functional calculus. No use is made of an elementary solution or of Green's function.

A number of other fixed point theorems and implicit function theorems have been proved for general spaces by topological methods. We remark that topological methods yield no information about the differentiability of functions defined implicitly.

Throughout the following paragraphs it is understood that the spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are complete normed linear spaces.

A very general fixed point theorem is as follows. Let f be a continuous transformation of a convex closed set S in the space $\mathfrak{X}$ into a part of itself, such that the transformed set $f(S)$ is compact. Then the transformation f leaves one point of S fixed. A fixed point theorem for n-dimensional space was proved by Brouwer. The proof of the general theorem depends on an extension of the Brouwer theorem, and is made by defining transformations f_n which approximate the transformation f on n-dimensional subsets of the set S. The first extension to function spaces was made by Birkhoff and Kellogg in 1922 [**3**]. Theorems for general spaces were proved later by Schauder [**25**].

The following theorem on inversion of functional transformations is readily proved. Let $y = F(x)$ be a transformation defined and continuous on an arcwise connected region X_0 of the space $\mathfrak{X}$, which is locally invertible, that is, the equation $y = F(x)$ has a unique continuous solution near each of its solutions. Suppose also that whenever the sequence $F(x_n)$ converges, the sequence

(x_n) is compact on X_0. Then F is a topological transformation of X_0 into the whole space $\mathfrak{Y}$. It is clear that a fixed point theorem for a transformation $G(x)$ may be obtained as a corollary by setting $F(x)=x-G(x)$. The theorem may be proved as follows. Let $Y_0=F(X_0)$, and let (y_n) be a sequence of points of Y_0 converging to a point y'. Then the corresponding sequence (x_n) is compact on X_0 and hence there is a subsequence (x_{n_k}) having a limit x' in X_0. Since F is continuous, $y'=F(x')$. Hence the set Y_0 is closed. It is also open, since the transformation is invertible near each of its points. Hence $Y_0=\mathfrak{Y}$. If the inverse transformation were not single-valued, there would be a curve B in X_0 joining two distinct points whose image C in the space $\mathfrak{Y}$ is a closed curve. By a continuous deformation of the curve C into a point the curve B must also be deformed into a point, which easily leads to a contradiction. Essentially this theorem and proof were given by Paul Levy in 1920 [**23**]. Caccioppoli has recently given an exposition of it in general form [**5**].

Schauder has recently proved theorems showing that under certain circumstances the uniqueness of the solution of an equation $y=G(x)$ implies the existence of a solution (x, y) for every y near y_0, where (x_0, y_0) is a solution. One of his theorems is as follows [**27**, p. 686]. Let there be defined in the space $\mathfrak{X}$ a weak convergence satisfying the postulates set down in Part I. Suppose also that every bounded set is weakly compact. Let F and H be transformations defined on an open set X_0 of $\mathfrak{X}$, and let F be totally continuous, in the sense that it transforms a weakly convergent sequence into a sequence converging according to the norm. Suppose that H satisfies a Lipschitz condition with constant $k<1$, and transforms a weakly convergent sequence into a weakly convergent sequence. Then if $G(x)\equiv x+F(x)+H(x)$ establishes a one-to-one correspondence between the open set X_0 and the transformed set $G(X_0)$, $G(X_0)$ is also an open set. Again the proof is made by constructing approximating transformations defined in subspaces of a finite number of dimensions. Schauder applied this theorem to show that in case a partial differential equation of elliptic type

$$F(x, y, z, p, q, r, s, t) = \psi(x, y), \qquad (z = \phi \text{ on the boundary}),$$

is known to have at most one solution, and if it has a solution for $\psi=\psi_0$, $\phi=\phi_0$, then it has a solution for ψ, ϕ near ψ_0, ϕ_0.

Another implicit function theorem has been given by Leray and Schauder [22] for equations of the form

$$x = F(x, k). \tag{13}$$

Let $\overline{R}$ be the closure of a bounded open set R in a space $\mathfrak{X}$, and let K be a finite closed segment of the real axis. Let the transformation F be defined and continuous on $\overline{R}K$, and continuous in k uniformly with respect to x. Moreover, let the set of functional values $F(\overline{R}, K)$ be compact, and suppose equation (13) has no solution with x on the boundary of R. Suppose that at an initial point k_0 of the interval K, the equation (13) has a finite number of solutions $x=x_j$, and that the sum of the "indices" of these solutions is not zero. This last hypothesis is satisfied, for example, in case there are an odd number of solutions near each of which the equation $y=x-F(x, k_0)$ has at most one solution. Then equation (13) has a continuum of solutions (x, k) on which k takes all the values of the interval K. This theorem has the advantage of being a theorem in the large, and of not requiring uniqueness of the initial solution, but the other hypotheses are rather stringent. The proof again depends on transformations F_ϵ which approximate the transformation F on subsets of R of a finite number of dimensions.

All of the theorems which have been quoted have their applications to boundary value problems for differential equations and to integral equations, both linear and non-linear. The field seems to be a fruitful one for further study.

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UNIVERSITY OF CHICAGO