ON THE LOCUS OF AN ANALYTIC EQUATION IN THE REAL PLANE*

BY A. B. BROWN

I have been unable to find in the literature a statement or proof of the following theorem.

THEOREM. *Let $f(x, y)$ be analytic at the real point (x_0, y_0), with $f(x_0, y_0)=0$ and $f(x, y)$ irreducible at (x_0, y_0).*† *Then the locus of the equation $f(x, y)=0$ in the real xy plane near (x_0, y_0) consists of one of the following three: (1) the point (x_0, y_0); (2) a single smooth curve through (x_0, y_0); (3) a cusp with vertex at (x_0, y_0).*

More detailed descriptions of (2) and (3) are contained in the proof which follows.

By a change of coordinates we may suppose $x_0=y_0=0$. According to the Weierstrass preparation theorem for the case of one independent variable, since f is irreducible at (0, 0), either $f(x, y)\equiv x\Omega(x, y)$, with $\Omega(0, 0)\neq 0$ and Ω analytic at (0, 0) or

$$(1) \qquad f(x, y) \equiv [y^m + A_1(x)y^{m-1} + \cdots + A_m(x)]\Omega(x, y),$$

with Ω as above, $m>0$, $A_j(x)$ analytic at $x=0$, and $A_j(0)=0$, $(j=1, \cdots, m)$. Since in the first case the real locus $f(x, y)=0$ is merely a straight line, it is sufficient to consider the case that (1) holds.

Since f is irreducible at (0, 0), the same is true of the algebroid function in (1), and hence its m-leaved Riemann surface is connected near (0, 0) and we can uniformize locally as follows:

$$(2) \qquad x = t^m,$$

$$(3) \qquad y = \psi(t) = a_1t + a_2t^2 + \cdots,$$

with ψ analytic at $t=0$, and a neighborhood of the origin in the

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† That is, not the product, near (x_0, y_0) in the 4-space of the complex variables, of two functions each analytic and zero at (x_0, y_0). For theorems which we use involving functions of complex variables, see W. F. Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, Chapter 8, §14, and vol. 2, part 1, Chapter 2, §§2, 4, 7, 9, 10, 11.

t plane giving exactly the points (x, y) in a neighborhood of $(0, 0)$ satisfying $f(x, y)=0$, each just once. Now x is real if and only if

$$(4) \qquad t = \tau e^{k\pi i/m}, \qquad (k = 0, 1, \cdots, m-1),$$

with τ real and positive, negative, or zero. Substituting from (4) in (3), for a fixed value of k we get

$$(5) \qquad y = \phi_k(\tau) = b_{k,1}\tau + b_{k,2}\tau^2 + \cdots, \qquad b_{k,\nu} = a_\nu e^{k\nu\pi i/m}.$$

Either all the coefficients in (5) are real, in which case *equation* (4) *for the given* k, *together with* (2) *and* (3), *gives only real points* (x, y), or else there is a first non-real $b_{k,j}$. In this case we have from (5) that

$$y = R(\tau) + \tau^j(b_{k,j} + b_{k,j+1}\tau + b_{k,j+2}\tau^2 + \cdots),$$

where $R(\tau)$ is real. Since $b_{k,j}$ is not real, we see that for sufficiently small τ the parenthesis is not real. Since τ is real, it follows that for small values of $\tau\neq 0$, y is not real. When $\tau=0$, $y=0$. We thus see that in this case *only* $\tau=0$ *gives real* y. Next we show that at most one value of k can give real y for real $\tau\neq 0$.

Suppose $t=\tau e^{k\pi i/m}$ and $t=\tau e^{j\pi i/m}$, with $0<j-k<m$, both gave real loci for real τ, when used with (2) and (3). Then $b_{k,\nu}=a_\nu e^{k\nu\pi i/m}$ and $b_{j,\nu}=a_\nu e^{j\nu\pi i/m}$ would be real, $(\nu=1, 2, \cdots)$, and hence if $a_\nu\neq 0$, their ratio $e^{(j-k)\nu\pi i/m}$ would be real. Therefore $(j-k)\nu/m$ would be an integer. Let $m=m_1\cdot m_2$, where m_1 is the H. C. F. of $(j-k)$ and m. Thus $m_2>1$, since $0<j-k<m$. Then ν would have to be a multiple of m_2, and hence

$$(6) \qquad y = \psi(t) \equiv \zeta(t^{m_2}),$$

with ζ analytic at the origin. Now if we set $t_1=te^{2\pi i/m_2}$, then $(t_1)^m=t^m e^{2\pi m_1 i}=t^m$, and $t_1{}^{m_2}=t^{m_2}e^{2\pi i}=t^{m_2}$. Hence from (2) and (6) we see that t_1 and t would give the same (x, y). But since $m_2>1$, $t_1\neq t$ if $t\neq 0$, and therefore we would have a contradiction. Consequently *at most one of the* m *values of* k *gives real* (x, y) *when used in* (2), (3), *and* (4) *with real* $\tau\neq 0$.

We now consider the case that some value of k does give a real locus. We define as the *branches* of the locus for the k in question the parts for which $\tau\geqq 0$ and $\tau\leqq 0$, respectively. Thus the two branches have the origin as a common end point. From (4), (2), and (3) we see that *on each of the two branches* y *is a* *single-valued real function of x, analytic when $x \neq 0$. If m is even, x has the same sign on each branch. If m is odd, x has opposite signs on the two branches except at $x=0$, and y is a single-valued real continuous function of x for x on an interval of the real axis containing the origin.*

From (2) and (3) we see that (i) if $a_1=a_2=\cdots=a_m=0$, then $\lim_{x\to 0} y/x=0$; (ii) if $a_1=\cdots=a_{m-1}=0$ but $a_m\neq 0$, then $\lim_{x\to 0} y/x=a_m\neq 0$; (iii) if some $a_j\neq 0$ with $0<j<m$, then when x approaches zero, y/x becomes infinite. Hence the two branches have a common tangent line. By a rotation of axes we now arrange that the x axis is the tangent line at the origin, and that $x\geqq 0$ on the real locus if m is even. Since $\lim_{x\to 0} y/x$ must then be 0, we must have case (i), so that $a_1=\cdots=a_m=0$. Then, for $t\neq 0$,

$$(7)\qquad \frac{dy}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)=[(m+1)a_{m+1}t^m+\cdots]\left(\frac{1}{mt^{m-1}}\right).$$

Therefore $\lim_{x\to 0} dy/dx=0$. Hence we have a *continuously turning tangent*; though if m is even, properly speaking we may only say that each branch has the positive x axis as right-hand tangent at the origin (a cusp).

Now from (7), for $t\neq 0$,

$$(8)\qquad \frac{dy}{dx}=b_{m+1}t+b_{m+2}t^2+\cdots.$$

Hence when $t\neq 0$,

$$(9)\qquad \frac{d^2y}{dx^2}=\left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]\cdot\frac{dt}{dx}=(b_{m+1}+2b_{m+2}t+\cdots)\left(\frac{1}{mt^{m-1}}\right).$$

Therefore on either branch, with $t\neq 0$, d^2y/dx^2 is either identically zero or nowhere zero. Hence *unless the locus is a straight line, d^2y/dx^2 has a fixed sign on each branch except at (x_0, y_0) where it may be zero, and the branch does not meet T, the tangent at (x_0, y_0), except at (x_0, y_0).* We also see that *as a point on the locus approaches (x_0, y_0), the curvature either approaches a fixed limit, possibly zero, or becomes infinite. The curvature at (x_0, y_0) exists for each branch and equals this limit* (also in the infinite case, where we have infinite curvature at (x_0, y_0)), as follows from (2), (8), and (9).

As examples we mention $y^3-x^4=0$, $y^3-x^5=0$, and $[y-1+(1-x^2)^{1/2}]^2-x^5=0$ with the determination of $(1-x^2)^{1/2}$ which equals 1 when $x=0$. These give respectively, at the origin, a minimum, a point of inflection, and a cusp with both branches concave upward. In none of the three cases is y analytic in x at the origin. An example where the locus is a single point is given by $y+ix=0$.

In the case of a reducible function $f(x, y)$, the real locus $f(x, y)=0$ neighboring (x_0, y_0) consists of a finite number of configurations of the kind described in the theorem, no two of which have any point except (x_0, y_0) in common. This is easily proved by use of theorems on resultants and on divisibility of one function by another. Of course two irreducible factors may have exactly the same locus.

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A PARTIAL DIFFERENTIAL EQUATION CONNECTED WITH THE FUNCTIONS OF THE PARABOLIC CYLINDER*

BY HARRY BATEMAN

The partial differential equation

$$\sum_{s=1}^{p}\left(\frac{\partial^2 V}{\partial x_s^2} - x_s\frac{\partial V}{\partial x_s}\right) + \nu V = 0, \tag{1}$$

which was considered by Mehler† in 1866, is a slight modification of an equation which occurs in wave-mechanics in the theory of the rotator in a plane and in space.‡ The case in which ν is a positive integer is then of chief physical interest and Mehler's simple solution

$$V = \prod_{s=1}^{p} H_{m_s}(x_s), \qquad \sum_{s=1}^{p} m_s = \nu, \tag{2}$$

acquires a physical significance. The function $H_m(x)$ is the polynomial of Laplace and Hermite defined by the equation

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† F. G. Mehler, Journal für Mathematik, vol. 66 (1866), p. 161.

‡ A. Sommerfeld, *Atombau und Spektrallinien, wellenmechanischer Ergänzungsband*, 1929, p. 23.