FOUR BOOKS ON GROUP THEORY AND QUANTUM MECHANICS

Gruppentheorie und Quantenmechanik. By Hermann Weyl. Second revised edition. Leipzig, S. Hirzel, 1931. xi+366 pp.

The Theory of Groups and Quantum Mechanics. By Hermann Weyl. Translated from the second revised German edition by H. P. Robertson. New York, E. P. Dutton, 1932. xxii+422 pp.

Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren. By Eugen Wigner. (Die Wissenschaft: Einzeldarstellungen aus der Naturwissenschaft und der Technik, Band 85, herausgegeben von Wilhelm Westphal.) Braunschweig, F. Vieweg und Sohn, 1931. viii+332 pp.

Die Gruppentheoretische Methode in der Quantenmechanik. By B. L. van der Waerden. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band 36.) Berlin, Springer, 1932. viii+157 pp.

The theory of groups has always played a role in the formulation and technical exploitation of mathematical theories of natural phenomena; but this role has often been merely a subordinate or an implicit one. Thus, for example, the analysis of space and time has always involved group-theoretic considerations; and classical dynamics has tended to an ever closer relation with the theory of groups through the transformation theory and the various related devices which play so important a part in the studies of Poincaré and Birkhoff. Nevertheless, these branches of mathematical physics have been advanced on the classical level without any essential use of the more profound portions of the theory of groups. In the transition to the radically altered theories of space, time, and motion which are currently accepted, the role of the theory of groups has been notably enhanced. This is true of the relativity theory and, to an even greater degree, of the more recently formulated quantum dynamics. Indeed, there are extensive portions of the quantum-theoretical discussion of atoms and molecules which would be only partially intelligible without some knowledge of the theory of the representations of an abstract group by means of linear transformations. Since the new physical theories spring from radical attempts to eliminate untenable epistemological postulates from the analysis of natural phenomena, it is not altogether surprising that they require more varied and more subtle contributions from mathematics (regarded as the appropriate instrument of abstract rational speculation) in proportion as they couple a wider range of physical fact with a narrower a priori basis of intuitive judgment.

In reviewing any work on the quantum theory, one is tempted to essay a brief statement of the fundamental epistemological principles which determine in broad outline the general mathematical structure of the whole theory. In the present instance such a statement would prove singularly valuable in enabling us to indicate in a quite precise fashion the essential role of group theory and even to forecast advances which may be expected from certain extensions of our present knowledge of group representations. While the task would not be a formidable one, it would carry us beyond the limits of a simple review and therefore cannot be undertaken here. Without having developed the necessary general background, we can do no more than consider, in somewhat vague terms, an important and instructive special case. For this purpose we select the atom. Familiar empirical evidence permits us to describe the atom as a compound dynamical system, built up of simpler interacting systems—the nucleus and a number of electrons—and characterized by certain properties of symmetry. The spatial symmetry about the nucleus and the essential likeness of the electrons are mathematically expressible in terms of the group of space rotations, $\mathfrak{d}_3$, and the group of permutations upon n objects, or symmetric group, π_n. Thus the theory of the groups $\mathfrak{d}_3$ and π_n must enter into the discussion of the atom, whatever may be the assumptions about the further properties of the constituent systems and their interactions. The physically observable wave characteristics of the electron, on the other hand, suggest the introduction of the mathematical apparatus for expressing undulatory phenomena as linear superpositions of elementary components. It is by virtue of this underlying linearity of wave theory that the linear representations of the groups $\mathfrak{d}_3$ and π_n have come to play so important a part in the quantum-theoretical treatment of the atom. Wigner was the first to observe this connection and to appreciate its significance as the essential key to the complex order of the atomic spectra. His fundamental paper appeared in the Zeitschrift für Physik (vol. 43 (1927), p. 624). The situation, however, is somewhat less simple than we have so far indicated. In order to arrive at a correct theory of the atom, it is necessary to make more precise the characteristics assigned to the electron. Before the development of the wave theory of the electron, Goudsmit and Uhlenbeck had already pointed out that the behavior of the electron could be brought into closer harmony with spectroscopic observations by ascribing to it an appropriate magnetic moment or spin. In a paper which appeared in the same volume of the Zeitschrift für Physik as Wigner's, on page 601, Pauli showed how the wave and spin properties of the electron could be combined. It is found that the resulting theory of the spinning electron involved the group $\mathfrak{u}_2$ of unimodular unitary transformations on two variables. Since the group $\mathfrak{u}_2$ is a double-valued representation of the group $\mathfrak{d}_3$, its representation theory includes that of $\mathfrak{d}_3$. The spinning electron may be introduced on the basis of relativistic arguments, as a natural consequence of the Lorentz-invariant wave equations of the electron discovered by Dirac (Proceedings of the Royal Society, (A), vol. 117 (1928), p. 610, and vol. 118 (1928), p. 351). We may remark in passing that, despite the successful union between relativity theory and quantum theory brought about in the case of the single electron, a complete reconciliation of the two theories has not yet been effected. The properties of the spinning electron are not alone sufficient to account for the behavior of the atom. Both chemical and spectroscopic evidence point to the stable arrangement of the electrons of an atom in shells about the nucleus. To describe the empirical data Pauli in 1925 introduced his famous exclusion principle. For the theory of the spinning electron it may be expressed in the form: no two electrons in an atom exist in the same state. Mathematically the exclusion principle assumes a quite different form: the description of the atom as a compound wave must be antisymmetric in the constituent spinning electrons. In the latter form one sees that the group π_n plays the dominating role. The relation of the group π_n and the exclusion principle to the study of the formation of molecules from atoms was developed by Heitler, London, Weyl, and Rumer, who showed that the phenomenon of chemical valence can be fully discussed on the basis of general theory. Thus, we can indicate the importance of the groups $\mathfrak{d}_3$, π_n, and $\mathfrak{u}_2$ in developing the theoretical structure of the atom in harmony with the chemical and spectroscopic data; but we must at the same time point out that there is nothing in our remarks that really goes to the heart of the matter or that ascribes to group theory any function except that of a very useful mathematical device in this connection. That the relation of group theory to the foundations of quantum mechanics is far more profound than we have been able to show in this illustration, is the thesis brilliantly maintained by Weyl in a paper (Zeitschrift für Physik, vol. 46 (1927), p. 1) and, more fully, in the book which is under review.

If this review had been more promptly written, it would surely have reflected in the extravagance of its phraseology the eager, almost feverish, interest displayed by physicists in the newly useful subject of group theory. For Wigner's fundamental observation suddenly confronted physicists with an unexpected need for the more profound portions of group theory and stimulated them to an enthusiastic study of a branch of mathematics which they had been inclined to neglect; and it was some little time before they were able to approach the critical task of selecting those parts of group theory which were to remain indispensable for their technical ends. The intervening years have brought the selective process to a point of temporary and approximate equilibrium, a state of affairs to which Weyl already refers in the preface to his second edition when he alludes to the feeling of relief experienced by many physicists over the elimination of the "group-pest" from their science. This elimination has been accomplished by virtue of several different circumstances: in the first place, the physicist may confine his attention to the particular groups which are important for his purposes, and these are, chiefly, the groups $\mathfrak{d}_3$, π_n, $\mathfrak{u}_2$ noted above and the groups occurring in crystallography; in the second place, the linear representations of many of these groups can be determined in ways already familiar independently of any general theory, as is the case with the representations of $\mathfrak{d}_3$ in terms of the spherical harmonics; and, in the third place, Slater's observation that the Pauli principle can be applied in the preliminary stages of theoretical discussion simply by writing down the necessary antisymmetric wave function avoids most of the complications arising from the symmetric group π_n. While one can easily understand the physicist's preference for special mathematics as opposed to the general and the abstract, one cannot help feeling that the elimination is quite possibly undesirable and surely not definitive. An attentive reading of Weyl's book or a careful consideration of the basic concepts of the quantum theory give grounds for believing that portions of the representation theory not yet well understood by mathematicians may prove indispensable in the further development of mathematical physics. Certainly, if this be the case, no theoretical physicist will have cause for regret if, by present attention to one of the most beautiful portions of abstract mathematics, he later finds himself prepared for one of those bewilderingly rapid advances which seem to be the lot of his subject.

Of the books before us, the most general in scope and the most profound in penetration is that of Weyl. As we have already remarked above, it represents the full and detailed elaboration of the ideas which he first set forth in a paper published in the Zeitschrift für Physik. Indeed, the paragraphs of this paper can be found scattered through the pages of the book. In main outline, the second edition now under review is like the first. There have been introduced many modifications and additions, enumerated in the foreword: in particular, much new material concerning the unification of the relativity and quantum theories has been incorporated in the second edition; and the difficult fifth chapter, which deals with the symmetric group and its application to the problem of valence, has been entirely revised. The scheme of presentation consists in the alternation of mathematical and physical material. The first chapter treats the geometry of unitary spaces, concluding with remarks on the generalization to Hilbert space (of which little or no use is made in the subsequent developments). The second chapter presents the general mathematical structure of quantum dynamics together with the perturbation methods and the discussion of many particular problems, such as the spectrum of the hydrogen atom and the interaction of the atom with radiation. Chapter 3 is devoted to the general theory of groups and their linear representations, with special attention to particular groups of interest for physical applications. Chapter 4 proceeds to apply the theory of groups to atomic spectra, the periodic table, the Lorentz-invariant wave equations of Dirac, and the Maxwell-Dirac field equations for quantum electrodynamics. Finally, the fifth chapter deals with the subject matter already noted above. The discussion of fundamental concepts and principles, whether mathematical or physical, is always stimulating and suggestive, conveying to the reader an extraordinarily vivid impression of the sweeping power of abstract mathematical thought. It must be said that the author's style, at once grandiloquent and concise, does not make easy the reading of the abstract material which he treats. To be sure, no abstract mathematical discussion can be fully appreciated without attentive study and some examination of concrete illustrations; but in the present instance the reviewer is persuaded that the general opinion regarding the difficulty of the book must be explained in part upon other grounds. Since this difficulty does not arise from ambiguity or lack of precision, the student who seriously desires to master this outstanding contribution to the mathematical literature of the quantum theory should not be deterred from doing so.

The translation of Weyl's book by Robertson was undoubtedly intended to render the great wealth of material which it contains more readily accessible to the student whose mother tongue is English. The reviewer's comments on the difficulty of reading the book now help to indicate the special value of the translator's service to English and American students; for many his work will help to reduce obstacles which, if not insurmountable, are at least very serious when met in a foreign language. The translation is essentially a literal one. It has been done painstakingly and well, in spite of a few confusing errors which have been brought to the reviewer's attention. Since some of these errors are not merely typographical in nature, we shall cite them here for the possible benefit of users to whose attention this review may come. On page 118, the sentence beginning on line 13 should read, "In this way, the elements of $\mathfrak{g}$ are distributed into sets of elements mutually equivalent with respect to $\mathfrak{h}$." On page 154, the sentence beginning on line 10 should read: "But in any case we can find a definite column index h with the following properties: there exist non-vanishing L-matrices whose first $h-1$ columns vanish; but all L-matrices whose first h columns vanish satisfy $L=0$." On page 156, the sentence beginning on line 9 from the bottom should read: "$\mathfrak{h}$, $\mathfrak{h}'$ being irreducible representations . . . ." It should be noted that the somewhat varied typographical problems have been well handled by the printer.

Wigner's book is much more elementary in character than that of Weyl. The development is clear and as concrete as the subject permits. Many simple illustrations are given so that the reader may test the content of the more abstract portions. The first three chapters are devoted to the theory of matrices. They are followed by three chapters concerning the fundamental concepts of quantum mechanics, together with perturbation methods. The discussion of groups and their linear representations, with special reference to those of primary interest to the physicist, occupies Chapters 7–16. The application of the general theory to the study of atomic spectra is given in the next three chapters, without the introduction of the spinning electron. The discussion of spin is carried through in Chapter 20, which serves as a foundation for the four last chapters on atomic spectra. While in some places a mathematician would express himself differently or even a little more precisely, Wigner has provided here one of the best introductory accounts of the subject. From the physicist's point of view it probably has the advantage of performing an inoculation with the germs of the "group-pest" in approximately the correct quantity. The book is less stimulating then Weyl's; and it does not attempt to cover so wide a range of subject matter, the relativity and valence problems being excluded. Of course, it should be recalled that the discussion of valence by group-theoretical methods was in its infancy at the time the book was published. On the other hand, Dirac's wave equations were as well known at that time as they are today. The author's omission of the Dirac theory can therefore be regarded as an exercise of his own judgment. In consequence of his decision in this matter, he has been compelled to give an original treatment of the spinning electron, based on Newtonian relativity alone, which the reviewer considers one of the most interesting and suggestive parts of the book.

The book of van der Waerden seems to be confined so far as possible to the exposition of group-theoretic technique. The physical discussions are mainly brief summaries of the necessary background material without emphasis on general concepts, or straightforward applications of the facts of group theory to physical problems. The six chapters deal with an introduction to quantum mechanics; groups and their representations, with special treatment of the rotation and Lorentz groups and discussion of simpler cases by way of illustration, followed by applications to atomic spectra; Dirac's equations and the spinning electron, with a more complete discussion of spectra; the permutation group and the Pauli principle; molecular spectra and the formation of molecules. The exposition is extraordinarily clear and compact. In particular, the treatment of group theory displays to the best advantage the simplification in technique obtainable through a proper use of abstraction. To the reviewer it seems that the book would be especially useful as a source and reference book in connection with discursive lectures. It could hardly be used as an introduction to the physical theory. While it does not contain an index, differing in this respect from all the other books under review, it is so brief that the table of contents is a quite effective guide to the specific subjects treated in the text.

For the student of the quantum theory, these books are indispensable if he wishes to master and to use the group-theoretical techniques appropriate to his problems. He will naturally wish to keep them on his shelves beside such well known theoretical works as Dirac's *Principles of Quantum Mechanics* and von Neumann's *Mathematische Grundlagen der Quantenmechanik*, in which the role of group theory is not emphasized. Mathematicians who value the fructifying contacts of their abstract realm with the more concrete world of physics will find this latest meeting both fascinating and inspiring; and those who interest themselves rather in the pure theory of groups will wish to consult the books of Weyl and van der Waerden both for their masterly exposition of the representation theory and for their suggestions of problems yet unsolved.

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A CORRECTION

A regretted slip was made in my review of Smith and Ginsburg's *History of Mathematics in America* (this Bulletin, vol. 41 (1935), pp. 603–606), in the list of suggestions for a new edition. Line 16, page 606—namely, p. 37, l. 14, *for* "of" *read* "on"—should be eliminated.

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