ASSOCIATED ALGEBRAIC AND PARTIAL DIFFERENTIAL EQUATIONS

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1. *Introduction.* Between an algebraic equation and an ordinary linear homogeneous differential equation with constant coefficients, in a single unknown, there is a familiar and useful association, illustrated by the equations

$$x^2 + 3x - 4 = 0, \qquad u'' + 3u' - 4u = 0.$$

Riquier* has shown that a similar association exists between finite systems consisting of m algebraic equations in n unknowns and m linear homogeneous partial differential equations in a single unknown and n independent variables as follows. Let

$$(1) \qquad \sum_{0}^{p} \overset{\alpha}{a}_{i_1\cdots i_n}(i_1 \cdots i_n) = 0, \qquad (\alpha = 1, \cdots, m),$$

be any system of algebraic equations, where the a's are constants and $(i_1 \cdots i_n)$ is to be interpreted as the monomial $x_1^{i_1} \cdots x_n^{i_n}$. Let the associated system of partial differential equations be

$$(2) \qquad \sum_{0}^{p} \overset{\alpha}{a}_{i_1\cdots i_n}(i_1 \cdots i_n)u = 0, \qquad (\alpha = 1, \cdots, m),$$

where $(i_1 \cdots i_n)$ is to be interpreted as the differential operator $\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$.

We shall prove in this paper the two following theorems.

THEOREM 1. *System* (1) *is inconsistent if and only if the general solution of* (2) *is* $u=0$.

THEOREM 2. *The general solution of* (2) *is a non-zero polynomial if and only if* $x_1 = \cdots = x_n = 0$ *is the solution of* (1).

2. *Corresponding Operations on the Two Systems.* We shall mul-

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* *Sur la résolution numérique du système d'équations algébriques entières à un nombre quelconque d'inconnues*, Annales Scientifiques de l'École Normale Supérieure, vol. 63 (1928), pp. 145–188.

tiply the equations by $(j_1 \cdots j_\nu)$ and then form linear combinations of them with constant coefficients. The interpretation of the $(j_1 \cdots j_\nu)$ is to be that of the system into which it is multiplied. This gives rise to new associated systems of equations.

3. *System* (1) *Inconsistent.* Let us assume (1) is inconsistent. Then according to a known theorem* there exist for system (1) C's satisfying the relation

$$1 \equiv \sum_{\alpha}^{1,\cdots,m} C^{\alpha}\left(\sum_{0}^{p} a^{\alpha}_{i_1\cdots i_n}(i_1 \cdots i_n)\right), \tag{3}$$

where

$$C^{\alpha} = \sum_{0}^{r} k^{\alpha}_{i_1\cdots i_n}(i_1 \cdots i_n). \tag{4}$$

The corresponding result on (2) would of course be obtained by multiplying both sides of the equations (3) on the right by u. Then (2) implies $u=0$ and from the form of (2) it is seen to be an actual solution.

If (1) has a solution $(\bar{x}_1, \cdots, \bar{x}_n)$, then (2) has the obvious *immediate*† solution $c \exp(\bar{x}_1x_1 + \cdots + \bar{x}_nx_n)$, where c is an arbitrary constant. On the assumption that $u=0$ is the general solution of (2), it follows immediately from this that (1) must be inconsistent. Thus Theorem 1 is established.

4. *System* (1) *Consistent with only the Trivial Solution.* Now consider the case where $x_1 = \cdots = x_n = 0$ is the solution of (1). According to Hilbert's zero theorem there exist C's similar to (4) and positive integers ρ_j such that

$$x_j^{\rho_j} \equiv \sum_{\alpha} C^{\alpha j}\left(\sum_{0}^{p} a^{\alpha}_{i_1\cdots i_n}(i_1 \cdots i_n)\right), \tag{5}$$

$$(\alpha = 1, \cdots, m;\ j = 1, \cdots, n).$$

Again, multiplying both sides of (5) on the right by u and interpreting $x_j^{\rho_j}$ as in (2), we see that the left member becomes $\partial^{\rho_j}u/\partial x_j^{\rho_j}$. System (2) implies that these derivatives vanish

* B. L. Van der Waerden, *Moderne Algebra*, vol. 1, 1930, p. 10.

† Riquier, loc. cit.

identically. Observe that there must be a non-zero solution of (2) else we have a contradiction with Theorem 1.

Consider any derivative $(i_1 \cdots i_n)$ whose order equals or exceeds $\rho_1 + \cdots + \rho_n$, that is,

$$(i_1 - \rho_1) + \cdots + (i_n - \rho_n) \geqq 0.$$

Then $i_j \geqq \rho_j$ for at least one j. Hence every derivative of the order considered is zero. Since there are but a finite number of derivatives below a given order taken with respect to a finite number of variables, it is clear that the series expansion solution for u reduces to a non-zero polynomial. If we assume that a non-zero polynomial is the general solution of (2), then (1) cannot have more than the trivial solution. For if it did, (2) would have a non-constant exponential solution. But if (1) has no solution our present hypothesis contradicts Theorem 1. Therefore (1) has the trivial solution and that alone, and Theorem 2 is proved.

We give an example illustrating the second result. Let (1) be

$$x_1^3 + x_2^3 = 0, \qquad x_1^3 - x_2^3 = 0,$$

and its associated system of partial differential equations (2) be

$$\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} = 0, \qquad \frac{\partial^3 u}{\partial x_1^3} - \frac{\partial^3 u}{\partial x_2^3} = 0.$$

Clearly $\rho_1 = \rho_2 = 3$. Thus the series expansion for u reduces to

$$u = \sum_0^2 a_{i_1 i_2} x_1^{i_1} x_2^{i_2},$$

where the coefficients are arbitrary constants.

An interesting application of the results obtained in this paper is that the conditions given by Riquier* for the existence of a solution of a system of algebraic equations (1) can now be extended to read: *The system* (1) *is consistent if and only if there exists a non-zero solution of* (2). In other words, *if* (2) *has any non-zero solution, it also has a non-zero immediate solution.*

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* Riquier, loc. cit.