TRANSFORMS OF FUCHSIAN GROUPS*

BY P. K. REES

This paper gives four theorems concerning the relative sizes of the isometric circles of the transformations, $T(z) = (az+\bar{c})/(cz+\bar{a})$, of a Fuchsian group and those of the transforms, $S(z) = GTG^{-1}(z) = (Az+\overline{C})/(Cz+\overline{A})$, of T in which $G(z) = (\alpha z+\bar{\nu})/(\nu z+\bar{\alpha})$ is considered as fixed and T any transformation of the Fuchsian group.

THEOREM 1. *The necessary and sufficient condition that the radii, r_s and r_t, of the isometric circles of S and T be equal is that the midpoint, $(a-\bar{a})/(2c) = m$, of the line segment joining the centers, g_t and g_t', of the isometric circles, I_t and I_t', of T and T^{-1} be on the circle $Q_5(z)$ with the origin and the center, $g = -\bar{\alpha}/\nu$, of the isometric circle of G as opposite ends of a diameter or on the circle $Q_6(z)$ with the origin and $1/\bar{g}$ as opposite ends of a diameter.*

PROOF. The equations of $Q_5(z)$ and $Q_6(z)$ are

$$Q_5(z) = 2\nu\bar{\nu}z\bar{z} + \alpha\nu z + \bar{\alpha}\bar{\nu}\bar{z} = 0, \quad Q_6(z) = 2\alpha\bar{\alpha}z\bar{z} + \alpha\nu z + \bar{\alpha}\bar{\nu}\bar{z} = 0.$$

If z lies on either Q_5 or Q_6, then $Q_5(z)Q_6(z) = 0$. But

$$(1)\quad \frac{1}{r_s^2} - \frac{1}{r_t^2} = -(a-\bar{a})(-\alpha\nu\bar{c} + \bar{\alpha}\bar{\nu}c) - \alpha\bar{\alpha}\nu\bar{\nu}[(a-\bar{a})^2 - 2c\bar{c}] - (\alpha\nu\bar{c})^2 - (\bar{\alpha}\bar{\nu}c)^2,$$

which vanishes if and only if $r_s = r_t$. Multiplying (1) equated to zero by $-[(a-\bar{a})/(2c\bar{c})]^2$ and replacing $(a-\bar{a})/(2c)$ by m, we have $Q_5(m)Q_6(m) \equiv 0$.

THEOREM 2a. *The necessary and sufficient condition that $r_s < r_t (r_s > r_t)$ is that $z = m$ substituted in the expression for Q_5Q_6 makes that expression positive (negative).*

PROOF. $r_s \lesseqgtr r_t$ according as $1/r_s^2 - 1/r_t^2 \gtreqless 0$. Furthermore

$$Q_5Q_6 = -\left(\frac{1}{r_s^2} - \frac{1}{r_t^2}\right)\frac{(a-\bar{a})^2}{4c^3\bar{c}^3}.$$

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Therefore $Q_5Q_6 \lesseqgtr 0$ according as $1/r_s^2 - 1/r_t^2 \lesseqgtr 0$, that is, according as $r_s \gtrless r_t$.

THEOREM 2b. *The necessary and sufficient condition that $r_s < r_t$ is that m be outside both Q_5 and Q_6 or inside both; the necessary and sufficient condition that $r_s > r_t$ is that m be inside Q_5 or Q_6 and outside the other.*

PROOF. The expressions for each Q_5 and Q_6 are negative (positive) according as m is inside (outside) the circle. Theorem 2b follows from this and Theorem 2a.

REMARK 1. The diameter of Q_5 is equal to $|g| = |\bar{\alpha}/\nu|$. This can be made as large as one may wish by choosing $|\nu|$ sufficiently near zero. Furthermore, the radius of Q_6 is the reciprocal of that of Q_5. Hence, by choosing G with $|\nu|$ sufficiently near zero, one can make the region inside Q_5 or Q_6 and outside the other as nearly a half-plane as desired. Therefore, for g sufficiently large, those transformations of the group T with m in approximately one half-plane (the one g is in) have their isometric circles increased in magnitude by transforming by G whereas those with m in the other approximate half-plane have $r_s < r_t$.

Furthermore by choosing $|g|$ sufficiently near to unity one can make the region inside Q_5 or Q_6 and outside the other as small as he may wish. Thus the transformations with m in as nearly the entire plane as desired have their isometric circles decreased in magnitude by transforming by G.

THEOREM 3. *The necessary and sufficient condition that $r_s = r_t/k$, k a non-negative real number, is that m lie on the locus*

$$(2) \quad (2\alpha\bar{\alpha}\nu\bar{\nu}z\bar{z} + \alpha\nu^2\bar{\nu}z + \alpha\bar{\alpha}^2\bar{\nu}\bar{z})(2\alpha\bar{\alpha}\nu\bar{\nu}z\bar{z} + \bar{\alpha}\nu\bar{\nu}^2\bar{z} + \alpha^2\bar{\alpha}\nu z) = k^2 z\bar{z}\alpha\bar{\alpha}\nu\bar{\nu}.$$

PROOF. From the definitions of r_s and r_t and from the equation $r_s = r_t/k$, we have $(r_t/r_s)^2 = (C\overline{C})/(c\bar{c}) = k^2$. Replacing $C\overline{C}$ by its value in terms of the coefficients of T and G and then replacing $(a-\bar{a})/(2c)$ by m, we have (2), since $c/\bar{c} = -\overline{m}/m$.

REMARK 2. The number k is not determined by (2) for a real, since then $m = 0$. However, m is on both Q_5 and Q_6 for $m = 0$, and therefore, by Theorem 1, $k = 1$.

COROLLARY 1. The absolute minimum value of k is zero; this value is taken on if the midpoint of the line segments (g_t, g_t') and $(g, 1/\bar{g})$ coincide and is possible only for T an elliptic transformation.

PROOF. Substituting $m = -(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$ into (2), we see that $k=0$ if $(a-\bar{a})/(2c) = -(\alpha\bar{\alpha}+\nu\bar{\nu})/(2\alpha\nu)$. Furthermore, we have $Q_0[-(\alpha\bar{\alpha}+\alpha\bar{\nu})/(2\alpha\nu)]>0$ for all G and all T of Fuchsian type, whereas $Q_0[(a-\bar{a})/(2c)]>0$ for T elliptic only.

REMARK 3. Changing (2) to trigonometric form, one finds the discriminant of the resulting quadratic in ρ to be

$$f(k) = 4(\alpha\nu e^{i\theta} + \bar{\alpha}\bar{\nu}e^{-i\theta})^2 - 16\alpha\bar{\alpha}\nu\bar{\nu}(1 - k^2).$$

This is a perfect square if and only if $k=1$ or 0; hence (2) is factorable rationally in terms of the coefficients of G in these two cases and only in them. The factors for $k=1$ are Q_5 and Q_6 of Theorem 1, and for $k=0$ they are immediate from (2).

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THE EQUATION $2^x - 3^y = d$*

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1. *Introduction.* According to Dickson's *History of the Theory of Numbers,*† Leo Hebreus, or Levi Ben Gerson (1288–1344), proved that $3^m \pm 1 \neq 2^n$ if $m>2$, by showing that $3^m \pm 1$ has an odd prime factor. The problem had been proposed to him by Philipp von Vitry in the following form: All powers of 2 and 3 differ by more than unity except the pairs 1 and 2, 2 and 3, 3 and 4, 8 and 9. In 1923 an elegant short proof by Philip Franklin appeared in the American Mathematical Monthly.‡

In 1918 G. Polya§ published a very general theorem which, as was later pointed out by S. Sivasankaranarayana Pillai,‖ proved as special cases that the equations

* Presented to the Society, October 26, 1935.

† Vol. 2, p. 731; see J. Carlebach, Dissertation, Heidelberg, 1909, pp. 62–64.

‡ Vol. 30 (1923), p. 81, problem 2927.

§ *Zur Arithmetische Untersuchung der Polynome,* Mathematische Zeitschrift, vol. 1 (1918), pp. 143–148.

‖ Journal of the Indian Mathematical Society, vol. 19 (1931), pp. 1–11.