A RECURSION FORMULA FOR THE POLYNOMIAL SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION*

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Let us denote by O the differential operator

$$\frac{\partial^n}{\partial x^n} - a_1\frac{\partial^n}{\partial x^{n-1}\partial y} - a_2\frac{\partial^n}{\partial x^{n-2}\partial y^2} - \cdots - a_n\frac{\partial^n}{\partial y^n},$$

where the a's are constants, real or complex. Define

$$\phi(\lambda) = \lambda^n - a_1\lambda^{n-1} - a_2\lambda^{n-2} - \cdots - a_n.$$

Let A be any nth order matrix having $\phi(\lambda)=0$ as its minimum equation, and define $Z=Ax+Iy$. For any function $F(Z)$,

$$\frac{\partial^n F}{\partial x^{n-t}\partial y^t} = \frac{d^n F}{dZ^n}A^{n-t}.$$

Then

$$OF = \frac{d^n F}{dZ^n}\phi(A),$$

and since $\phi(A)=0$, $F(Z)$ is a solution of $OF=0$. If we can write

$$F(Z) = f_1(x, y)I + f_2(x, y)A + \cdots + f_n(x, y)A^{n-1},$$

we have

$$OF(Z) = \sum_{i=1}^{n} Of_i(x, y)A^{i-1} = 0.$$

Since A satisfies no equation of degree $n-1$,

$$Of_i(x, y) = 0 \qquad (i = 1, 2, \cdots, n).$$

These results are due to Spampinato.†

By a proper choice of the matrix A, this theorem of Spampinato leads to an interesting recursion formula for the poly-

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nomial solutions of $Of(x, y) = 0$ by means of which all such solutions can be computed with rapidity.

Let

$$A = \left\| \begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{matrix} \right\| .$$

Then $\phi(\lambda) = 0$ is the minimum equation of A. By means of the relation $\phi(A) = 0$, $(Ax + Iy)^k$ can be written as a linear combination of $I, A, A^2, \cdots, A^{n-1}$. The first rows of these matrices are, respectively,

$$(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), (0, 0, 1, \cdots, 0), \cdots, (0, 0, 0, \cdots, 1).$$

Hence if we denote the first row of $(Ax + Iy)^k$ by $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$, we have

$$(Ax + Iy)^k = p_1^{(k)} I + p_2^{(k)} A + \cdots + p_n^{(k)} A^{n-1}.$$

If we define the vector

$$v_k = (p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}),$$

it follows from the relation

$$(Ax + Iy)^{k+1} = (Ax + Iy)^k (Ax + Iy),$$

that

$$v_{k+1} = v_k (Ax + Iy),$$

or in full,

$$\begin{aligned} p_1^{(k+1)} &= y p_1^{(k)} && + x a_n p_n^{(k)}, \\ p_2^{(k+1)} &= x p_1^{(k)} + y p_2^{(k)} && + x a_{n-1} p_n^{(k)}, \\ p_3^{(k+1)} &= \qquad x p_2^{(k)} + y p_3^{(k)} && + x a_{n-2} p_n^{(k)}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && \\ p_{n-1}^{(k+1)} &= x p_{n-2}^{(k)} + y p_{n-1}^{(k)} && + x a_2 p_n^{(k)}, \\ p_n^{(k+1)} &= x p_{n-1}^{(k)} && + (y + x a_1) p_n^{(k)}, \end{aligned} \tag{1}$$

where $p_1^{(1)} = y$, $p_2^{(1)} = x$, $p_i^{(1)} = 0$, $i > 2$.

It is evident that $p_i^{(k)}$ is a homogeneous polynomial of degree k, and by Spampinato's theorem, it is a solution of $Of=0$.

For $k \geqq n$ there are just n linearly independent homogeneous polynomials of degree k which satisfy $Of=0$. For if we let

$$P_k = A_0 x^k + A_1 x^{k-1} y + \cdots + A_k y^k$$

where the A's are to be determined, then $\partial^n P_k / \partial x^n$ will have the form

$$h_1 A_0 x^{k-n} + h_2 A_1 x^{k-n-1} y + \cdots + h_k A_{k-n} y^{k-n},$$

where each h is a positive integer. By forming OP_k, we obtain a homogeneous polynomial of degree $k-n$ which must vanish identically. This condition leads to $k-n+1$ equations in the $k+1$ unknown A's. Since the matrix of the coefficients

$$\begin{Vmatrix} h_1 & * & * & \cdots \\ 0 & h_2 & * & \cdots \\ 0 & 0 & h_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{Vmatrix}$$

is obviously of rank $k-n+1$, there are exactly n linearly independent solutions.

For $k<n$, the components of v_k are, in order, the terms in the binomial expansion of $(y+x)^k$ followed by zeros. Hence, for $k<n$, every homogeneous polynomial of degree k is a linear combination of $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$.

We shall prove by induction that for $k \geqq n-1$,

$$p_i^{(k)} = c_{ik} y^{k-i+1} x^{i-1} + \cdots, \qquad (i = 1, 2, \cdots, n), \tag{2}$$

where the c_{ik} are positive integers and the terms omitted are of degree less than $k-i+1$ in y. For $k=n-1$, (2) follows from (1). If we assume (2) to hold for some $k \geqq n-1$, we have from (1)

$$\begin{aligned} p_i^{(k+1)} &= x p_{i-1}^{(k)} + y p_i^{(k)} + x a_{n-i+1} p_n^{(k)} \qquad (p_0^{(k)} = 0) \\ &= x c_{i-1,k} y^{k-i+2} x^{i-2} + y c_{ik} y^{k-i+1} x^{i-1} \\ &\quad + x a_{n-i+1} c_{nk} y^{k-n+1} x^{n-1} + \cdots \\ &= (c_{i-1,k} + c_{ik}) y^{k-i+2} x^{i-1} + \cdots, \end{aligned}$$

where the terms represented by $\cdots$ are of degree less than $k-i+2$ in y. Thus (2) is established for $k+1$ and hence by induction for $k \geqq n-1$. Hence $p_1^{(k)}, p_2^{(k)}, \cdots, p_n^{(k)}$ are linearly independent for $k \geqq n-1$.

We have now proved that, for every k, every linear homogeneous polynomial of degree k which is a solution of $Of=0$ has the form

$$c_1 p_1^{(k)} + c_2 p_2^{(k)} + \cdots + c_n p_n^{(k)},$$

where the c's are arbitrary constants.

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A CHARACTERIZATION OF NULL SYSTEMS IN PROJECTIVE SPACE

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1. *Introduction.* We consider the n-dimensional projective space S_n defined analytically by means of any abstract field F. The points P of S_n are given by a set of $n+1$ elements x_i of F, $P=(x_0, x_1, \cdots, x_n)$, (not all $x_i=0$), with the convention that proportional sets define the same point. The points P whose coordinates satisfy a linear homogeneous equation $u^{(0)}x_0+u^{(1)}x_1+\cdots+u^{(n)}x_n=0$, (not all $u^{(i)}=0$), form a hyperplane $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$. There is no difficulty in defining such notions as those of straight lines, projections, and cross ratios, and discussing the elementary properties.

Let M be a non-singular skew-symmetric bilinear form with coefficients a_{ik} in F,

$$M = \sum_{i,k=0}^{n} a_{ik} y_i x_k, \qquad a_{ik} = -a_{ki}, \qquad \det(a_{ik}) \neq 0.$$

For every point $P=(x_0, x_1, \cdots, x_n)$ the equation $M=0$ is the equation of a hyperplane ϵ in the coordinates $(y_0, y_1, \cdots, y_n)$ of a variable point of ϵ. We obtain in this manner a one-to-one correspondence between the points $P=(x_0, x_1, \cdots, x_n)$ and hyperplanes $\epsilon=(u^{(0)}, u^{(1)}, \cdots, u^{(n)})$ of S_n which is called a *null system*. The relation between corresponding values of the $u^{(i)}$ and x_i is given by