applied to the example discussed in these lines, is unknown. In fact it is not clear that there exist sets E for which $H(E)$ is not vacuous. But it is obvious that $H(E)$ is always contained in $K(E)$, where K is the set-function (assumed additive) in terms of which H is defined.

In the example in question it is true that an "accessible" topology can be defined in terms of neighborhoods in such a way that the function Lx_n defined in terms of these neighborhoods is identical with the original function L, and so that the set of all continuous functions is dense on the whole space.

INSTITUTE FOR ADVANCED STUDY

ON (2, 2) PLANAR CORRESPONDENCES

BY L. H. CHAMBERS

1. *Introduction.* Most of the existing literature dealing with (2, 2) planar transformation is of the type given by the product of two harmonic homologies. By this I mean that the pairs of points of the plane π (or π'') are in harmonic homology. Papers of this type were given by E. Amson,* T. Kubota,† and P. Visalli.‡ Barraco§ defined an involutorial (2, 2) transformation of the plane by means of an involution between the tangents to a conic from points of the plane.

In this paper I shall consider only periodic (2, 2) transformations of period two. The treatment in each case, except those involving the Bertini involution, will be analytic. A synthetic treatment of some of the cases has been given by Sharpe and Snyder.|| I shall use the following theorems proved in their paper.

A necessary and sufficient condition that the two images of a point P describe distinct loci as P moves on a curve C is that C touches the branch curve at every non-fundamental point they have in common.

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* Erlangen Dissertations, vol. 130 (1903–04).

† Science Reports, Tôhoku, vol. 6 (1918), and vol. 14 (1925).

‡ Circolo Matematico di Palermo, Rendiconti, vol. 3 (1889), pp. 165.

§ Giornale di Matematiche, vols. 53–54 (1915–16).

|| Transactions of this Society, vol. 18 (1918), pp. 409.

A necessary and sufficient condition that a (2, 2) transformation be the product of a (2, 1) transformation and a (1, 2) transformation is that the defining curves of the one plane (and hence of the other) define a net.

Bertini proved that every rational involution of the plane was one of four types, namely, harmonic homology, Geiser, Jonquière, or Bertini.* Castelnuovo showed† that these four involutions could be mapped on a double plane and that planar involutions of any order are always rational.

I shall define the (2, 2) transformations of this paper as follows. Consider any of the transformations (H), (G), (J), (B) as existing in the planes π and π'', and mapped doubly upon a plane π' in such a manner that the two points in involution correspond to a single point of the plane π'. Associating points of π' with pairs of points in involution of the planes π and π'' will define a (2, 2) transformation.

The (2, 2) transformation, as defined, is periodic and of period two. A point P_1 of the plane π has for its image in π' a point P_1', and P_1' has for its image in π'' two points P_1'', P_2'', which are in involution. By the inverse transformation the points P_1'', P_2'', have for image the point P_1' of π'. The image of P_1' in π is the point P_1, and a point P_2, associated with P_1 in the involution of the plane π.

By this method of generating (2, 2) transformations, there are sixteen types. Of these, only ten are distinct. Various cases of each type arise due to the mapping used in the plane π'. Superposition of fundamental elements of the plane π' cause a reduction in the order of the transformations.

Zeuthen's theorem‡ will not apply in the Bertini types if the image curve, in the plane of the Bertini, degenerates unless both components of the curve are considered simultaneously. The mapping of (H), (G), and (J) upon a double plane has been done by Snyder§ and will not be repeated.

2. *The Mapping of* (B) *Upon a Double Plane.* This will be done by a different method than previously employed. The (B)

* Annali di Matematica, (2), vol. 8 (1877), p. 244.

† Rendiconti dei Lincei, (5), vol. 2 (1893), p. 205.

‡ F. Severi, *Trattato di Geometria Algebraico*, vol. 1, part 1, pp. 209.

§ V. Snyder, this Bulletin, vol. 30, pp. 101–124 (1920).

involution of π can be defined by the web of curves

$$a\phi^2 + b\phi\psi + c\psi^2 + df = 0, \tag{1}$$

where ϕ and ψ are general cubic curves and f is a sextic having double points at eight of the nine intersections of ϕ and ψ.*

Refer the points of π to those of a 3-way space by the equations

$$\xi = \phi^2, \qquad \eta = \phi\psi, \qquad \zeta = \psi^2, \qquad \tau = f. \tag{2}$$

The pairs of points in involution are mapped doubly upon the cone $\Gamma \equiv \eta^2 - \xi\zeta = 0$ whose vertex arises from the ninth point of intersection of ϕ and ψ and whose generators correspond to cubics of the pencil $\Lambda \equiv \phi + \lambda\psi = 0$. By a stereographic projection† of Γ upon π', (B) will be mapped doubly upon π'.

The inverse transformation is obtained by solving the equations

$$\begin{aligned} \psi\xi - \phi\eta &= 0, \\ f\zeta - \psi^2\tau &= 0. \end{aligned} \tag{3}$$

Since sixteen roots of this solution are known, the resulting equation is quadratic. The coincidence curve is

$$K_9 \equiv \frac{\partial(\phi, \psi, f)}{\partial(x_1, x_2, x_3)} = 0. \tag{4}$$

We have $K_9 : 8Q_i^3$. A cubic of Λ meets K_9 in three variable points, hence K_9 is represented in the 3-way space as the intersection of Γ with a cubic surface and projects into $L_6' : (P_1' \equiv P_2')^3$ (that is, two consecutive 3-fold points). The tangent γ' to L_6' at $P_1' \equiv P_2'$ is determined by the tangent plane to Γ at the vertex of projection.

A line $l(x)$ is met by any cubic Λ in three points and is represented in π' by a $C_6' : (P_1' \equiv P_2')^3$, $4P'^2$. $8Q_i \sim 8C_2' : (P_1' \equiv P_2')$ with γ' as tangent. Points of γ' correspond to directions through P_1, P_2, the images of the vertex of projection. The line $l'(x')$ meets each fundamental conic in two points and has for image a $C_6 : 8Q_i^2$, P_1, P_2. A line $l'(x') : (P_1' \equiv P_2')$ will determine two generators of Γ, one of which is fixed; hence $(P_1' \equiv P_2') \sim C_3$ of Λ.

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* V. Snyder, American Journal of Mathematics, vol. 33 (1910), p. 43.

† Snyder and Sisam, *Analytic Geometry of Space*, p. 145.

3. *The Transformation* $(H)(H)$. Let the two transformations (H), of the planes π and π'', be mapped independently upon the plane π'.

CASE I. If the transformation (x') into $(\bar{x}')$ is

$$x_i' = \sum_{j=1}^{3} g_{ij}\bar{x}_j',$$

the (2, 2) transformation is obtained by combining the two (2, 1) transformations and this (1, 1) transformation. The F-points of π are (0, 1, 0) and $(\pm(G_{11}G_{31})^{1/2}, G_{21}, G_{31})$, which $\sim g_{31}x_1''^2 + g_{32}x_2''x_3'' + g_{33}x_3''^2 = 0$ and $x_3'' = 0$, respectively. L_4 is a degenerate quartic composed of two conics. A similar F system and branch curve exist in π''. The line $l(x) \sim C_4' : [(0, 1, 0) \equiv (0, 1, 0)]^2$. The line $l''(x'')$ has a similar image in π.

CASE II. If the transformation existing between (x') and $(\bar{x}')$ is $x_1' = \bar{x}_i'$, the (2, 2) transformation is rational. There are no F-elements in either π or π'', and $L_1 \equiv x_1 = 0$, $L_1'' \equiv x_1'' = 0$. The line $l(x) \sim C_2''$ composed of two lines. The line $l''(x'')$ has a similar image in π.

4. *The Transformation* $(G)(H)$. Combining the mapping of (G) of π with the mapping of (H) of π'', the resulting (2, 2) transformation is of the type (G) (H). $7Q_i \sim 7C_2'$. $P_1, P_2 \sim x_3'' = 0$. (0, 1, 0) of $\pi'' \sim x_1C_2 - x_2C_3 = 0$. $L_6 \equiv (x_2C_3 - x_3C_2)(x_1C_2 - x_2C_3) = 0$. $L_8' : [(0, 1, 0) \equiv (0, 1, 0)]^4$. The line $l(x) \sim C_6'' : P_1''^2, P_2''^2$ $[(0, 1, 0) = (0, 1, 0)]^4$, $2P''^2$. The line $l''(x'') \sim C_6 : 7Q_i^2$.

5. *The Transformation* $(J)(H)$. By combining the mapping of (H) of π'' with the mapping of (J) of π, the (2, 2) transformation (J) (H) is obtained.

CASE I. Let the transformation of the plane π' be

$$x_i' = \sum_{j=1}^{3} g_{ij}\bar{x}_j.$$

(0, 0, 1) of $\pi \sim C_{2(m-1)}'' : [(0, 1, 0) \equiv (0, 1, 0)]^{m-1}$, $2P''^{m-2}$. P_1, $P_2 \sim x_3'' = 0$. $4(m-1)Q_i \sim 4(m-1)C_2'' : (0, 1, 0)$. P_3, $P_4 \sim x_3'' = 0$. (0, 1, 0) of $\pi'' \sim C_{m+1} : (0, 0, 1)^{m-1}$. P_1'', P_2'' $\sim M_1 - a_3M_2 = 0$. P_3'', $P_4'' \sim a_1x_1 + a_2x_2 = 0$. $L_{2(m+1)}$ degenerates into two parts of order $(m+1)$, each part having an $(m-1)$-fold point at (0, 0, 1). $L_{4m}'' : [(0, 1, 0) \equiv (0, 1, 0)]^{2m}$, $2P^{2(m-1)}$, $2P^2$.

The line $l(x) \sim C''_{2(m+1)}: [(0, 1, 0) \equiv (0, 1, 0)]^{m+1}$, $2P^m$. The line $l''(x'') \sim C_{2(m+1)}: (0, 0, 1)^{2(m-1)}$, $4(m-1)Q_i^2$, P_1^2, P_2^2.

CASE II. Let the transformation of the plane π' be $x_i' = \bar{x}_i'$. $4(m-1)Q_i \sim 4(m-1)C_2''$. P_1, $P_2 \sim C_2''$. $(0, 0, 1) \sim C''_{2(m-1)}$ $: [(0, 0, 1) \equiv (0, 0, 1)]^{m-2}$, $[(0, 1, 0) \equiv (0, 1, 0)]^{m-1}$. P_3, P_4 $\sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim M_2 = 0$. $(0, 0, 1)$ of $\pi'' \sim M_1 - a_3 M_2 = 0$. P_1'', $P_2'' \sim a_1 x_1 + a_2 x_2 = 0$. $L_{2(m+1)} \equiv x_1(a_1 x_1 + a_2 x_2)(M_1 - a_3 M_2) M_2$. $L''_{4m}: [(0, 1, 0) \equiv (0, 1, 0)]^{2m}$, $[(0, 0, 1) \equiv (0, 0, 1)]^{2(m-1)}$. The line $l(x) \sim C''_{2(m+1)}: [(0, 1, 0) \equiv (0, 1, 0)]^{m+1}$, $[(0, 0, 1) \equiv (0, 0, 1)]^m$. The line $l''(x'') \sim C_{2(m+1)}: 4(m-1)Q_i^2$, P_1^2, P_2^2, $(0, 0, 1)^{2(m-1)}$.

6. *The Transformation* (B) (H). CASE I. Let (H) of π'' be mapped upon π' and (B) of π be mapped upon π', so that $P_1' \equiv P_2'$ is a general point (a', b', c'). Eight points $Q_i \sim 8C_4''$ $: [(0, 1, 0) \equiv (0, 1, 0)]^2$. P_1, $P_2 \sim C_2'': (0, 1, 0)$. $P_3 P_4 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_6: 8Q_i^2$, P_1, P_2. $L_{12}: C_6 \cdot \overline{C}_6$, each of the net (1). $L_{12}'': 2(P_1'' \equiv P_2'')^2$, $[(0, 1, 0) \equiv (0, 1, 0)]^6$. The line $l(x) \sim C_{12}''$ $: [(0, 1, 0) \equiv (0, 1, 0)]^6$, $8P''^2$, $2(P_1'' \equiv P_2'')^3$. The line $l''(x'')$ $\sim C_{12}: 8Q_i^4$, P_1^2, P_2^2.

CASE II. Let (B) be mapped on π' so that $P_1' \equiv P_2' = (0, 1, 0)$, $\gamma' \neq x_1' = 0$, $\gamma' \neq x_3' = 0$. $8Q_i \sim 8C_4'': (0, 1, 0)^3$. $P_1 P_2 \sim C_2''$. $P_3 P_4 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_3$ of Λ. $L_6: 2C_3$ of Λ. L_{12}'' has 9 branches through $(0, 1, 0)$ by threes in three directions, $p'' = 10$. The line $l(x) \sim C_{12}: 8P''^2$ and the singularities of L_{12}''. The line $l''(x'') \sim C_{12}: 8Q_i^2$, P_1^2, P_2^2.

CASE III. Let (B) be mapped upon π' so that $P_1' \equiv P_2'$ $= (0, 1, 0)$, $\gamma' \equiv x_3' = 0$. $8Q_i \sim 8C_4'': (0, 1, 0)^3$ with $x_3'' = 0$ as tangent. P_1, $P_2 \sim x_3'' = 0$. $(0, 1, 0)$ of $\pi'' \sim C_3$ of Λ. $L_6: 2C_3$ of Λ. L_{12}'' has 9 branches through $(0, 1, 0)$ with $x_3'' = 0$ as tangent. The line $l(x) \sim C_{12}'': 8P''^2$ and singularities of L_{12}''. The line $l''(x'') \sim C_{12}: 8Q_i^4$, P_1^2, P_2^2.

7. *The Transformation* (G) (G). Combining two mappings of (G) we obtain the transformation $(G)(G)$. $7Q_i \sim 7C_3''$. $L_{12}: 7Q_i^4$. The line $l(x) \sim C_9'': 7Q_i^3$, $2P''^2$. Similar results hold for the plane π''.

8. *The Transformation* (G) (J). Let the transformation (J) of π'' be mapped upon π'. Combining this mapping with that of (G), we obtain the transformation (G) (J). $7Q_i \sim 7C''_{m+1}: P_1''$, P_2'', $4(m-1)Q_i''$, $(0, 0, 1)^{m-1}$. P_1, $P_2 \sim M_1 - a_3 M_2 = 0$. P_3, $P_4 \sim a_1 x_1''$

$+a_2x_2''=0$. $4(m-1)Q_i''\sim 4(m-1)C_3{:}\,7Q_i$. P_1'', $P_2'\sim C_3{:}\,7Q_i$. $(0, 0, 1)$ of $\pi''\sim C_{3(m-1)}{:}\,7Q_i^{m-1}$, $2P^{m-2}$. $L_{6m}{:}\,2P^{2(m-1)}$, $2P^2$, $7Q_i^{2m}$. $L''_{4(m+1)}{:}\,4(m-1)Q_i^{\prime 4}$, $P_1''^4P_2''^4$, $(0, 0, 1)^{4(m-1)}$. The line $l(x)\sim C''_{3(m+1)}{:}\,4(m-1)Q_i^3$, $P_1''^3$, $P_2''^3$, $(0, 0, 1)^{3(m-1)}$, $2P^2$. The line $l''(x'')\sim C_{3(m+1)}{:}\,7Q_i^{m+1}$, $2P^m$.

9. *The Transformation* $(B)(G)$. Let the transformation (G) of π'' be mapped upon π'. Combining this mapping with that of §2, we have the transformation $(B)(G)$. $8Q_i\sim 8C_6'\,{:}\,7Q_i''^2$. P_1, $P_2\sim C_3''{:}\,7Q_i'$. $7Q_i''\sim 7C_6{:}\,8Q_i^2$, P_1, P_2. $L_{24}{:}\,8Q_i^8$, P_1^4, P_2^4. $L_{18}''{:}\,2(P_1''\equiv P_2'')^3$, $7Q_i^6$. The line $l(x)\sim C_{18}''{:}\,8P''^2$ and singularities of L_{18}''. The line $l''(x'')\sim C_{18}{:}\,2P^2$, $8Q_i^6$, P_1^3, P_2^3.

10. *The Transformation* $(J)(J)$. Suppose the equations for the mapping of (J) of π'' upon π' are

$$x_1'x_3' - x_3'x_1'' = 0, \qquad x_2'N_1'' - v'N_2'' = 0,$$

where N_1'', N_2'' are curves of degree n having an $(n-2)$-fold point at $(0, 1, 0)$ and $v'=\sum_1^3 b_ix_i'$. Combining this mapping with that of (J) of π', we have the transformation $(J)(J)$. $4(m-1)Q_i\sim 4(m-1)C''_{n+1}{:}\,4(n-1)Q_i''$, P_1'', P_2'', $(0, 1, 0)^{n-1}$. P_1, $P_2\sim C''_{n+1}{:}\,(0, 1, 0)^{n-1}$, $4(n-1)Q_i''$, P_1'', P_2''. $(0, 0, 1)$ of $\pi\sim C''_{(m-1)(n+1)}{:}\,4(n-1)Q_i''^{m-1}$, $P_1''^{m-1}$, $P_2''^{m-1}$, $(0, 1, 0)^{(m-1)(n-1)}$, $2P''^{m-2}$. $2Q\sim N_1''-b_2N_2''$. $2Q\sim b_1x_1'+b_3x_3''=0$. Similar images exist in π for the F-system of π''. $L_{2n(m+1)}{:}\,2P^{2(n-1)}$, $2P^2$, $4(m-1)Q_i^{2n}$, P_1^{2n}, P_2^{2n}, $(0, 0, 1)^{2n(m-1)}$. $L''_{2m(n+1)}$ is similar to $L_{2n(m+1)}$. The line $l(x)\sim C_{(m+1)(n+1)}{:}\,2P''^m$, $4(n-1)Q_i''^{m+1}$, $P_1''^{m+1}$, $P_2''^{m+1}$, $(0, 1, 0)^{(n-1)(m+1)}$. A similar image exists for $l''(x'')$.

11. *The Transformation* $(B)(J)$. Case I. Let (B), of π, be mapped upon π' so that $P_1'\equiv P_2'=(a', b', c')$. $8Q_i\sim 8C_{2(m+1)}$ ${:}\,P_1''^2P_2'^2$, $4(m-1)Q_i'^2$, $(0, 0, 1)^{2(m-1)}$. P_1, $P_2\sim C''_{m+1}{:}\,P_1''$, P_2'', $4(m-1)Q_i$, $(0, 0, 1)^{m-1}$. $2Q\sim M_1''-a_3M_2''$. $2Q\sim a_1x_1''+a_2x_2''$. $4(m-1)Q_i''\sim 4(m-1)C_6$ of (1). P_1'', $P_2''\sim C_6$ of (1). $(0, 0, 1)$ of $\pi''\sim C_{6(m-1)}{:}\,8Q_i^{2(m-1)}$, P_1^{m-1}, P_2^{m-1}, $2P^{m-2}$. $2Q_i''\sim C_3$ of Λ $L_{12m}{:}\,8Q_i^{4m}$, P_1^{2m}, P_2^{2m}, $2P^2$, $2P^{2(m-1)}$. $L''_{6(m+1)}{:}\,(0, 0, 1)^{6(m-1)}$, $P_1'^6$, $P_2''^6$, $4(m-1)Q_i'^6$, $2(P_3''\equiv P_4'')$. $l(x)\sim C''_{6(m+1)}{:}\,8P''^2$ and the singularities of $L''_{6(m+1)}$. $l''(x'')\sim C_{6(m+1)}{:}\,8Q_i^{2(m+1)}$, P_1^{m+1}, P_2^{m+1}, $2P^m$.

CASE II. Map (B) upon π' so that $P_1' \equiv P_2' = (0, 0, 1)$. $8Q_i \sim 8''_{(m+2)}: P_1'', P_2'', 4(m-1)Q_i'$, $(0, 0, 1)^m$. $P_1, P_2 \sim C_1''$ $:(0, 0, 1)$. $2Q \sim a_1x_1'' + a_2x_2''$. One C_3 of $\Lambda \sim M_1'' - a_3M_2''$. $4(m-1)Q_i''$, P_1'', $P_2'' \sim (4m-3)$ C_3 of Λ. $(0, 0, 1)$ of $\pi'' \sim C_{3m}$ $:8Q_i^m$. $L_{6(m+1)}: 8Q_i^{2(m+1)}$, P_1^2, P_2^2, $2P^2$. $L''_{3(m+2)}: P_1'^3$, $P_2''^3$, $4(m-1)Q_i''$, $(0, 0, 1)^{3m}$, $2(P_3'' \equiv P_4'')^3$. $l(x) \sim C''_{3(m+2)}: 8P''^2$ and the singularities of $L''_{3(m+2)}$. $l''(x'') \sim C_{3(m+2)}: 8Q_i^{m+2}$, P_1, P_2. The complete image must be used to apply Zeuthen's theorem.

12. *The Transformation* (B) (B). CASE I. Let the two mappings of (B) upon π' be such that no F-elements coincide. $8Q_i \sim 8C_{12}'': 8Q_i''^4$, $P_1''^2$, $P_2''^2$. P_1, $P_2 \sim C_6'': 8Q_i''^2$, P_1'', P_2''. $2Q \sim C_3''$ of Λ''. $L_{36}: 8Q_i^{12}$, P_1^6, P_2^6, $2(P_3 \equiv P_4)^3$. A similar F-system and branch curve exist in π''. $l(x) \sim C_3'': 8P''^2$ and the singularities of L_{36}''. $l''(x'') \sim C_{36}$ similar to C_{36}''.

CASE II. Let the two mappings of (B) be such that $(P_1' \equiv P_2') = (\overline{P}_1' \equiv \overline{P}_2')$, $\gamma' \neq \bar{\gamma}'$. $8Q_i \sim 8C_9'': 8Q_i''^3$, P_1'', P_2''. $P_1P_2 \sim C_3''$ of Λ''. One C_3 of $\Lambda \sim C_3''$ of Λ''. $L_{27}: 8Q_i^9$, P_1^3, P_2^3. A similar F-system and branch curve exist in π''. $l(x) \sim C_{27}'': 8P''^2$, $8Q_i^9$, P_1^3, P_2^3. The complete image must be used to apply Zeuthen's theorem.

CASE III. Let the two mappings of (B) be such that $(P_1' \equiv P_2') = (\overline{P}_1' \equiv \overline{P}_2')$, $\gamma' = \bar{\gamma}'$. $8Q_i \sim 8C_6'': 8Q_i''^2$, P_1'', P_2''. P_1, P_2 $\sim P_1''$, P_2''. $L_{18}: 8Q_i^6$. $L_{18}': 8Q_i''^6$. A similar F-system exists in π''. $l(x) \sim C_{18}^2: 8Q_i''^6$, $8P^2$. $l''(x'') \sim C_{18}: 8Q_i^6$, $8P^2$.

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