THE CONTINUOUS ITERATION OF REAL FUNCTIONS*

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1. *Continuous Iterations.* Let $E(x)$ be a real, continuous, steadily increasing function of x in the range $-\infty < a \leqq x < \infty$ such that

$$E(x) > x, \qquad (x \geqq a), \tag{1}$$

and let $E_1(x) = E(x)$, $E_2(x) = E(E_1(x))$, $\cdots$ denote its successive iterates. In a previous note in this Bulletin, referred to hereafter as Note, one of us† has developed a simple formula for continuously iterating the function $E(x)$. We propose here to determine *all* continuous iterations of $E(x)$ subject to a restriction to be explained presently.

By a continuous iteration of $E(x)$ we shall understand a real function $\Theta_y(x)$ of the two real variables x and y with the following two properties

(i) $$\Theta_0(x) = x, \qquad \Theta_1(x) = E(x), \quad (x \geqq a).$$

(ii) $$\Theta_{y+z}(x) = \Theta_y(\Theta_z(x)), \qquad (x \geqq a,\ y,\ z \geqq 0).$$

The restriction which we shall impose upon the functions $\Theta_y(x)$ is the following:

(iii) *$\Theta_y(a)$ is a steadily increasing continuous function of y in the range $0 \leqq y \leqq 1$.*

2. *Prior Investigations.* The continuous iteration of real functions was discussed in detail by A. A. Bennett.‡ So far as the authors are aware, other investigators have confined their attention to the continuous iteration of analytic functions.§ The functional equation (ii) was first considered by A. Korkine,‖ who

* Presented to the Society, February 29, 1936.

† Ward, *Note on the iteration of functions of one variable*, this Bulletin, vol. 40 (1934), pp. 688–690.

‡ Annals of Mathematics, (2), vol. 17 (1916), pp. 23–69.

§ See the references in the Note.

‖ Bulletin des Sciences Mathématiques, (2), vol. 6 (1882), part 1, pp. 228-242.

proved formally a result equivalent to the first theorem of this paper, assuming that $\Theta_y(x)$ was differentiable with respect to y.

A complete discussion of the functional equation $E_n(x)=x$ with x, $E(x)$ real, n a positive integer, has been given by J. F. Ritt,* and W. Chayoth† has recently proved certain very general existence theorems on functional equations in the real domain.

3. THEOREM 1. *Any function Θ satisfying the conditions* (i), (ii), *and* (iii) *is continuous and steadily increasing in both x and y. Moreover for each such function $\Theta=E_y(x)$ there exists a unique, continuous, steadily increasing solution $\psi=f(x)$ of the functional equation*

$$(2) \qquad \psi(x+1) = E(\psi(x)), \qquad \psi(0) = 0$$

such that‡

$$(3) \qquad E_y(x) = f(f^{-1}(x) + y), \qquad (x \geqq 0,\ y \geqq 0).$$

We have taken here and throughout the remainder of the paper, $a=0$ and $E(a)=1$ as was shown to be possible without loss of generality in the Note.

To prove this theorem, let $\Theta=E_y(x)$ be a particular function satisfying the conditions (i), (ii), and (iii). Since $E_{x+1}(0)=E(E_x(0))$, we see from (iii) that $E_x(0)$ is continuous and steadily increasing in the range $0\leqq x<\infty$.

Write $f(x)$ for $E_x(0)$. Then $f(x)$ has a unique, continuous, steadily increasing inverse $f^{-1}(x)$ in the range $0\leqq x<\infty$ such that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x, \qquad f(0) = f^{-1}(0) = 0.$$

Also $f(x+y)=E_{x+y}(0)=E_y(E_x(0))=E_y(f(x))$. Hence $f(x+1)=E(f(x))$, and $\psi=f(x)$ is a solution of (2). Then

$$E_y(x) = E_y(f\{f^{-1}(x)\}) = f(f^{-1}(x) + y),$$

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* Annals of Mathematics, (2), vol. 17 (1916), pp. 113–122. See also the note by A. A. Bennett, loc. cit., p. 123.

† Monatshefte für Mathematik und Physik, vol. 39 (1932), pp. 279–288.

‡ The *converse* of this theorem is well known. See, for example, A. A. Bennett, Annals of Mathematics, volume cited, pp. 74–75; pp. 23–30.

which is (3). It is now evident that $E_y(x)$ is a continuous and steadily increasing function both of x and of y.

Finally, the function $f(x)$ in formula (3) is uniquely determined by $E_y(x)$. For letting $x=y=0$, and using (1), we see that $f(0)=0$. Hence $f^{-1}(0)=0$. Therefore on letting $x=0$ and $y=x$, $f(x)=E_x(0)$. The problem of determining all continuous iterations of $E(x)$ is thus reduced to the solution of the functional equation (2).

4. THEOREM 2. *Let $\theta(x)$ be a continuous function of x in the interval $0\leqq x<1$, which increases steadily from $\theta(0)=0$ to $\theta(1-0)=1$. Then every continuous steadily increasing solution ψ of the functional equation* (2) *is of the form*

$$(4) \qquad \psi(x) = E_{[x]}(\theta(x - [x])),$$

where $[x]$ denotes the greatest integer in x.

Conversely, for every such choice of $\theta(x)$, (4) *gives a continuous steadily increasing solution of the functional equation* (2).

First of all, every such increasing solution ψ of (2) tends to infinity with x. For assume that $\psi(x)$ tends to a finite limit L as $x\to\infty$. Then $\psi(x)<L$ for all finite values of x. Now by (1), $E(L)>L$. Hence, since $E(x)$ is continuous, there exists a positive number δ such that $E(L-\delta)>L$. Choose x_0 so that $\psi(x)>L-\delta$, $x\geqq x_0$. Then $\psi(x+1)=E(\psi(x))>E(L-\delta)>L$, giving a contradiction.

It follows that in the interval $0\leqq x<\alpha$, $\psi(x)$ has a unique, continuous, steadily increasing inverse $\phi=\phi(x)=\psi^{-1}(x)$ such that $\phi\to\infty$ as $x\to\infty$. This inverse is readily seen to satisfy the famous functional equation of Abel,*

$$(5) \qquad \phi(E(x)) = \phi(x) + 1, \qquad \phi(0) = 0.$$

For convenience, write e_n for $E_n(0)$, $(n=0, 1, 2, \cdots)$. Then $e_0=0$, $e_1=1$, and since by (1), $E(x)>x$, it follows that $e_n<e_{n+1}$.

We shall now show that $e_n\to\infty$. For otherwise, e_n tends to a finite limit k, and $e_n<k$, $(n=0, 1, 2, \cdots)$. Since $k>1$, if $E_{-1}(x)$ denotes the inverse of $E(x)$, then $E_{-1}(k)=M$, where $0<M<k$. For $k=E(E_{-1}(k))=E(M)>M$. Hence for all sufficiently large n, $e_n>M$. But then $e_{n+1}=E(e_n)>E(M)=k$, giving a contradiction.

* Works, vol. 2, Posthumous Papers, 1881, pp. 36–39.

It follows that, given any positive value of x, we can determine an integer k such that

$$e_k \leqq x < e_{k+1}. \tag{6}$$

Let x lie in the interval (6). Then from the properties of $E(x)$ and its ordinary iterates, we can write

$$x = E_k(y), \qquad y = E_{-k}(x), \qquad (0 \leqq y < 1),$$

where $E_{-k}(x)$ denotes the inverse of $E_k(x)$ in the interval $e_k \leqq x < \infty$.

Now in the interval $0 \leqq x < 1$, let us write $\theta^{-1}(x)$ for $\phi(x)$. Then $\theta^{-1}(0) = 0$, and $\theta^{-1}(x)$ increases steadily and continuously as x increases, and $\theta^{-1}(1-0) = \lim_{x\to 1}\theta^{-1}(x) = 1$ by (5). Furthermore, the inverse of $\theta^{-1}(x)$, which we denote by $\theta(x)$, exists and has the properties stated in Theorem 2. From (5), we see that

$$\phi(x) = \phi(E_k(y)) = \phi(y) + k = \theta^{-1}(y) + k$$

or

$$\phi(x) = \theta^{-1}(E_{-k}(x)) + k. \tag{7}$$

Since $0 \leqq E_{-k}(x) < 1$, we observe also that $k = [\phi]$, the greatest integer in $\phi(x)$.

To determine ψ, we need only solve (7) for x in terms of ϕ. We have

$$\phi - [\phi] = \theta^{-1}(E_{-[\phi]}(x)),$$

$$\theta(\phi - [\phi]) = E_{-[\phi]}(x),$$

$$x = E_{[\phi]}(\theta(\phi - [\phi])).$$

Hence*

$$\psi(x) = E_{[x]}(\theta(x - [x])). \tag{4}$$

The proof of the converse for a function θ satisfying the conditions of the theorem is almost word for word the same as in the special case $\theta(x) = x$, which has been given in full in the Note.

The function $\theta(x)$ is arbitrary save for the restrictions stated in the theorem. Once chosen, it fixes the iteration completely; it is in fact $E_x(0)$.

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* It is obvious that if $\pi(x)$ denotes a periodic function of x with period one, such that $\pi(x) = \theta(x)$, $(0 \leqq x < 1)$, then we can write $\psi(x) = E_{[x]}(\pi(x))$, or more concisely still, $\psi(x) = E_{[x]}(\psi(x - [x]))$, since $\psi(x - [x]) = \theta(x - [x])$.