ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

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1. *Introduction.* A set $\phi_0(x)=1$, $\phi_1(x)$, $\phi_2(x)$ of polynomials of degrees 0, 1, 2, $\cdots$, is called a set of orthogonal polynomials if they satisfy

$$\int_a^b \phi_m(x)\phi_n(x)d\psi(x) = 0, \qquad \int_a^b d\psi(x) > 0, \qquad (m \neq n),$$

where $\psi(x)$ is a non-decreasing function of bounded variation. There is no restriction in assuming the highest coefficient 1.

It has been shown by W. Hahn† that if the derivatives also form a set of orthogonal polynomials, then the original set were Jacobi, Hermite, or Laguerre polynomials. His method consisted in showing that the polynomials satisfy a differential equation of the type

$$(a + bx + cx^2)\phi_n'' + (d + ex)\phi_n' + \lambda_n\phi_n = 0.$$

From this it followed that the set were Jacobi, Hermite, or Laguerre polynomials.

Here we propose to give a new proof of this result, our point of view being to answer the question: What conditions on the weight function result from assuming that both $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are sets of orthogonal polynomials? However, we shall assume that (a, b) is a finite interval and $d\psi(x)=p(x)dx$,‡ where the weight function is L-integrable.

2. *A Relation for the Weight Function* $q(x)$. Let the set $\{\phi_n'(x)\}$ be orthogonal in the interval (c, d), infinite or not, with the weight function $q(x)$, that is,

$$\int_c^d q(x)\phi_n'(x)\phi_m'(x)dx = 0, \qquad (m \neq n).$$

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* Presented to the Society, April 11, 1936.

† W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

‡ Professor Shohat informs me that he has discussed the general case $d\psi(x)$.

The polynomials $\{\phi_n(x)\}$, $\{\phi_n'(x)\}$ satisfy the recurrence relations,*

$$\phi_{n+2}(x) = (x - c_{n+2})\phi_{n+1}(x) - \lambda_{n+2}\phi_n(x),$$

$$\frac{1}{n+2}\phi_{n+2}'(x) = \frac{x - c_{n+2}'}{n+1}\phi_{n+1}'(x) - \lambda_{n+2}'\phi_n'(x),$$

$$(n \geqq 0;\ c_n,\ c_n',\ \lambda_n,\ \lambda_n' \text{ constants}).$$

Differentiating both sides of the first relation and eliminating the term containing x, by means of the second relation, we get

$$\phi_{n+1}(x) = \frac{1}{n+2}\phi_{n+2}'(x) + c_{n+2}''\phi_{n+1}'(x) + \lambda_{n+2}''\phi_n'(x),$$

$$(c_n'',\ \lambda_n'' \text{ constants}).$$

Remembering that $\phi_n'(x)$, with the weight function $q(x)$, is orthogonal to any polynomial of degree $\leqq n-2$, we get

$$(1) \qquad \int_c^d q(x)\phi_{n+1}(x)G_{n-2}(x)dx = 0,$$

where $G_n(x)$ is an arbitrary polynomial of degree $\leqq n$.

LEMMA. *Let $Q(x)$ be non-negative in (c, d), and such that the numbers*

$$\beta_k = \int_c^d Q(x)x^k dx, \qquad (k = 0, 1, \cdots),$$

exist, and for a certain positive integer r

$$(2) \quad \int_c^d Q(x)\phi_n(x)G_{n-r-1}(x)dx = 0, \quad (n = r+1, r+2, \cdots).$$

Then almost everywhere

$$Q(x) = \begin{cases} P_r(x)p(x) & \text{in } (a, b), \\ 0 & \text{elsewhere}, \end{cases}$$

where $P_r(x)$ is a polynomial of degree $\leqq r$.

Consider the function

* J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, p. 24.

$$R(x) = (u_0 + u_1x + u_2x^2 + \cdots + u_rx^r)p(x).$$

We determine the $\{u_i\}$ so that

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx, \qquad (i = 0, 1, \cdots, r);$$

that is, the u_i satisfy the equations

$$\begin{aligned} \alpha_0u_0 + \alpha_1u_1 &+ \cdots + \alpha_ru_r = \beta_0, \\ \alpha_1u_0 + \alpha_2u_1 &+ \cdots + \alpha_{r+1}u_r = \beta_1, \\ &\cdots\cdots\cdots\cdots \\ \alpha_ru_0 + \alpha_{r+1}u_1 &+ \cdots + \alpha_{2r}u_r = \beta_r, \end{aligned}$$

where

$$\alpha_k = \int_a^b x^k p(x)dx.$$

This is always possible for the determinant of this system is known to be positive.* Now†

$$0 = \int_a^b R(x)\phi_{r+i}(x)dx = \int_c^d Q(x)\phi_{r+i}(x)dx, \qquad (i = 1, 2, \cdots).$$

And then

$$\int_a^b R(x)x^i dx = \int_c^d Q(x)x^i dx, \qquad (i = 0, 1, 2, \cdots).$$

Let

$$f(x) = \begin{cases} Q(x) - R(x) \text{ in } E_1, \text{ the points where } (a, b) \text{ and } (c, d) \text{ overlap,} \\ -R(x) \text{ in } E_2, \text{ the remainder of } (a, b), \\ Q(x) \text{ in } E_3, \text{ the remainder of } (c, d). \end{cases}$$

Then

$$\int_{E_1+E_2+E_3} f(x)x^i dx = 0, \quad (i = 0, 1, 2, \cdots).$$

If (c, d) is finite, $f(x)$ must be zero almost everywhere, and our

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* See Shohat, loc. cit., p. 9, formula (19).

† As a special case, take in (2), $n = r+i$; that is, $\int_c^d Q(x)\phi_{r+i}(x)G_{i-1}(x)dx = 0$, then take $G_{i-1}(x) \equiv 1$.

lemma follows. Moreover, from the above definition of $f(x)$, we conclude that $R(x)$, hence $p(x)$, $\equiv 0$ almost everywhere in E_3, so that both intervals (a, b) and (c, d) may be reduced to their common part E_1; in other words, here we may take $(c, d)\equiv(a, b)$. If (c, d) is infinite, let $A=\max(|a|, |b|)$, so that $|x|\leqq A$ in E_1+E_2, which is identical with (a, b), and let E_4 be the part of E_3 for which $|x|\geqq(1+\alpha)A$, where $\alpha>0$ is arbitrary. If $Q(x)\geqq 0$, we have

$$(1+\alpha)^i A^i \int_{E_4} Q(x)dx \leqq \int_{E_3} Q(x)x^i dx = \left| \int_{E_1+E_2} f(x)x^i dx \right|$$

$$\leqq A^i \int_{E_1+E_2} |f(x)|\, dx,$$

$$(1+\alpha)^i \leqq \frac{\int_{E_1+E_2} |f(x)|\, dx}{\int_{E_4} Q(x)dx},$$

for all even i, which is impossible unless $Q(x)$ is zero almost everywhere in E_4. Hence (c, d) reduces to $E_1+E_3-E_4$ and in (c, d), $|x|<(1+\alpha)A$; that is, in view of the arbitrariness of $\alpha(>0)$,

$$|x| \text{ in } (c, d) \leqq A = \max(|a|, |b|),$$

which requires that $(c, d)\equiv(a, b)$. The following important result has been established: (c, d) is finite and coincides with (a, b).

With this lemma, it follows from (1) that

$$q(x) = (rx^2 + sx + t)p(x), \tag{3}$$

and we may take $c=a$, $d=b$.

3. *Existence of* $q'(x)$. Consider the function

$$S(x) = k\int_a^x (x-l)p(x)dx,$$

where k and l are such that $S(b)=0$, $\int_a^b S(x)dx=\int_a^b q(x)dx$. An integration by parts applied to $\int_a^b S(x)\phi_{n+1}'(x)dx$ gives, since $S(a)=S(b)=0$,

$$\int_a^b S(x)\phi_{n+1}'(x)dx = \int_a^b \phi_{n+1}(x)k(x-l)p(x)dx = 0, \qquad (n \geqq 1).$$

But $q(x)$ is the weight function for the orthogonal polynomials $\{\phi_n'(x)\}$, whence

$$\int_a^b S(x)\phi_{n+1}'(x)dx = \int_a^b q(x)\phi_{n+1}'(x)dx = 0, \quad (n \geqq 1).$$

This and the relation $\int_a^b S(x)dx = \int_a^b q(x)dx$ gives

$$\int_a^b S(x)x^n dx = \int_a^b q(x)x^n dx, \qquad (n \geqq 0),$$

and then $q(x) = S(x)$ almost everywhere. Since $S(x)$ has a derivative almost everywhere, $q(x)$ has a derivative almost everywhere and

$$(4) \qquad q'(x) = k(x-l)p(x), \qquad q(a) = q(b) = 0.$$

4. *Discussion of $q(x)$ and $p(x)$.* Dividing (4) by (3), we get

$$(5) \qquad \frac{q'(x)}{q(x)} = \frac{k(x-l)}{rx^2+sx+t}, \qquad q(a) = q(b) = 0.$$

We proceed to show (i) rx^2+sx+t has real zeros, (ii) $r\neq 0$. (i) Assume rx^2+sx+t has imaginary zeros. Integrating the differential equation (5), we get

$$\log q(x) = \int \frac{k(x-l)}{rx^2+sx+t}dx + c,$$

$$q(x) = K(rx^2+sx+t)^\alpha e^{\beta \arctan(\gamma x+\delta)},$$

$$(\alpha, \beta, \gamma, \delta, K \text{ constants}).$$

This is incompatible with $q(a) = q(b) = 0$. (ii) Assume first $r = s = 0$. Equation (5) becomes

$$q'(x) = (2\alpha x + \beta)q(x),$$

$$q(x) = Ke^{\alpha x^2+\beta x}, \qquad (\alpha, \beta, K \text{ constants}),$$

which is not zero at a and b.

Next, suppose $r = 0$, $s\neq 0$. Equation (5) gives

$$\frac{q'(x)}{q(x)} = \frac{k(x-l)}{sx+t} = \alpha + \frac{\beta s}{sx+t},$$

$$q(x) = K(sx+t)^\beta e^{\alpha x}, \qquad (\alpha, \beta, K \text{ constants}).$$

This cannot vanish at both end points, $x=a$ and $x=b$.

Having thus proved (i) and (ii), we set

$$rx^2 + sx + t = r(x - g)(x - h), \qquad (r \neq 0,\ g,\ h \text{ real}),$$

and rewrite (5) as follows:

$$\frac{q'(x)}{q(x)} = \frac{k(x - l)}{rx^2 + sx + t} = \frac{\alpha}{x - g} + \frac{\beta}{x - h},$$

whence

$$q(x) = K(x - g)^{\alpha}(x - h)^{\beta}, \qquad (K,\ \alpha,\ \beta,\ \text{constants}).$$

The conditions $q(a)=q(b)=0$ demand that $g=a$, $h=b$, so that finally (disregarding inconsequential constant factors)

$$q(x) = -\, r(x - a)^{\alpha}(b - x)^{\beta}.$$

And then from (3)

$$p(x) = \frac{q(x)}{rx^2 + sx + t} = \frac{r(x - a)^{\alpha}(b - x)^{\beta}}{r(x - a)(b - x)},$$

$$p(x) = (x - a)^{\alpha-1}(b - x)^{\beta-1},$$

and we can see that α, β are both >0.

5. *Conclusion.* Since this is the weight function for Jacobi polynomials, we have thus established the following theorem.

THEOREM. *If* $\{\phi_n(x)\}$ *is a set of orthogonal polynomials with the weight function* $p(x)$ *in the finite interval* (a, b), *and if we assume that the derivatives* $\{\phi_n'(x)\}$ *also form a set of orthogonal polynomials in a certain interval* (c, d) *(infinite or not), with a non-negative weight function* $q(x)$, *then* $\{\phi_n(x)\}$ *is a set of Jacobi polynomials.*

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