ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

256. Professor C. M. Cramlet: *Linear differential equations with constant coefficients.*

First order differential equations are solved by reducing the equations to canonical form by elementary matrix operations. The method is elementary and practical for the solution of the homogeneous and non-homogeneous systems. (Received May 7, 1936.)

257. Professor C. M. Cramlet: *On the reduction of a representation to classical canonical form.*

A representation or mixed tensor a_s^i transforms in accordance with the matrix equations $\bar{a}=paq$, $pq=1$. The transformation matrix p such that a is transformed to canonical form $\bar{a}$ is found by simultaneously choosing p_i and q_i as elementary matrices, transforming a to canonical form by a succession of transformations. The method is practical for the reduction of a numerical matrix a. (Received May 7, 1936.)

258. Professor H. P. Robertson: *Geometry and physical space-time.*

There are two roles which geometry, or its four-dimensional extension, kinematics, may assume in physical science: (1) an *a priori* form, prerequisite to a quantitative description of the physical world, and (2), a *contingent* form, conditioned by the physical content of space-time. The following topics are treated: Influence on the former position, as exemplified by Newton and Kant, of the classical non-euclidean geometries of Gauss, Bolyai, Lobachévski, and Riemann; mathematical and philosophical development of this view by Helmholtz, Lie, and Russell. Extension to the four-dimensional space-time manifold in the special theory of relativity by Einstein and Minkowski, and in the later theories of de Sitter, Whitehead, and Milne. Development of the second position, envisaged by Riemann, in the general theory of relativity, by Einstein and Weyl; applications to cosmology by Friedmann, Lamaître, Tolman, and the author. Interfusion of the second position with the methods of the first in the cosmological applications of the author's kinematical investigations, and

the relationship thereto of the theories of Milne and Page. (Received May 7, 1936.)

259. Professor L. H. McFarlan: *Special forms of the Euler differential equations when x is absent from the integrand.*

In this paper the special form $f-y'f_{y'}=c$ assumed by the Euler differential equation for the most simple problem of the calculus of variations, when the integrand is free of x, is seen to follow in a direct manner if the problem is treated in the parametric form. It may also be considered as a problem of Lagrange with the initial end point fixed while the other end point is constrained to lie on a line parallel to the t axis. The auxiliary differential equation is $x'-1=0$. The transversality condition at the variable end point yields the special form of the Euler equation. This method of treatment lends itself readily to problems in which derivatives of order higher than the first enter into the integrand. (Received May 11, 1936.)

260. Professor J. P. Ballantine: *Arc cotangent triads.*

An arc cotangent triad is defined as a set of three numbers, the sum of whose arc cotangents vanishes; for example, (1, 1, 0), (−1, 2, 3), (−2, 3, 7). If one such triad (a, b, c) is known, then obviously $(-a, -b, -c)$, (b, a, c), and less obviously $(-a, b+2a, c+2a)$ are also. This leads to an interesting group of transformations generated by five self-reciprocal operations. It is proved that all integral arc cotangent triads can be generated from (1, 1, 0) by means of this group. (Received May 15, 1936.)

261. Dr. F. B. Jones: *Concerning more general topologically flat spaces.*

Suppose that S is a space satisfying R. L. Moore's axioms 0-4 and the following axiom: If P is a point of a region R, there exists in R a connected domain D containing P whose boundary is a subset of a collection Δ of mutually exclusive continua lying in $R-D$ such that no component of $D-P$ has a limit point in more than one element of Δ. In this paper the author shows that a number of plane analysis situs theorems hold in S and that if S is completely separable, S is a subset of a plane or a sphere. (Received May 18, 1936.)

262. Dr. F. B. Jones: *Certain equivalences and subsets of a plane.*

In a space satisfying R. L. Moore's axioms 0-4 and the author's axiom 5_1 (see abstract 39-9-254) a number of plane analysis situs theorems are shown to be equivalent. An example of a completely separable space satisfying these axioms is given in which none of these theorems hold true. This example also refutes the author's statement in the above abstract that such a completely separable space is a subset of a plane or a sphere. However, conditions are discussed which would make such a space a subset of a plane or a sphere. (Received May 18, 1936.)

263. Mr. W. H. Ingram: *On the forces occurring in a dynamical system where the coordinate axes have variable relative obliquity.* Preliminary report.

When the relative obliquity of a system of coordinate axes varies with the time, corresponding forces arise in the equations of motion of the dynamical system whose coordinates are referred to such axes. Application is made to the theory of commutator motors where the coordinate axes are rotating and, under certain conditions, mutually oblique with variable obliquity. (Received May 18, 1936.)

264. Professor L. I. Neikirk: *Some symbolic identities.*

Given an algebraic identity $\phi_1(y) = \phi_2(y)$; if y is replaced by the differentiation symbol D, we derive an operational identity $\phi_1(D) = \phi_2(D)$. When this is applied to a second identity $F_1(x) = F_2(x)$, the result will be a third identity. Most identities obtained in this way are easily obtained by other methods. It has, however, yielded valuable results in differential equations, invariants and covariants, and theory of numbers. Among the cases are Cayley's hyperdeterminates, Aronhold's symbolic notation in invariants and covariants, and Blissard's umbral notation in the theory of numbers. The author uses the method to expand the inverse operator and obtains the solution of the general linear differential equation of the first order as an infinite series of iterated integrals which are reduced to the conventional form. (Received May 19, 1936.)

265. Dr. R. D. James: *An extension of two formulas of Hurwitz.*

There are two formulas due to Hurwitz for the number of representations of the square of an arbitrary integer as a sum of three or five squares. In this paper an analogous formula for seven squares is proved. In addition it is shown that, in certain cases, no further formulas of this type are possible. (Received May 19, 1936.)

266. Dr. Max Zorn: *On a theorem of Lie.*

Given a linear family $(x_1, \cdots, x_k)$ of matrices over a non-modular field such that (a) $xy - yx$ is in the family; (b) every matrix in the family is nilpotent; then the theorem of Lie asserts that every matrix in the associative ring generated by the x_i is nilpotent, and that moreover any product of sufficient length in this ring vanishes. For this theorem a new proof is given in the following fashion: I. If symbols x_i are commutative, then each product of x_i is expressible as a sum of powers. II. The same is true if the x_i have the property (a). III. Hence in the ring in question any element is a sum of nilpotent elements. IV. Such a ring is necessarily nilpotent. (Received May 20, 1936.)

267. Dr. Max Zorn: *Nilpotency of finite groups.*

If a, b are elements of a finite group G, denote $aba^{-1}b^{-1}$ by $\langle a, b\rangle$. For two subgroups U, V let $\langle U, V\rangle$ be the subgroup generated by the $\langle u, v\rangle$. We prove the theorem: If all iterated commutators $\langle a\langle a\langle a \cdots \langle ab\rangle \rangle \rangle \rangle$ of sufficient length

are identity, then the group G is nilpotent, that is $\langle G\langle G\langle G \cdots \langle G, G\rangle\rangle\rangle\rangle$ is identity. The analog of this theorem in the theory of continuous groups is the theorem of Lie treated in the foregoing abstract. (Received May 20, 1936.)

268. Mr. I. E. Highberg: *A note on pseudo-polynomials in abstract spaces.*

In a previous communication (see abstract 42-5-136) I discussed the definition of pseudo-polynomials, gave a number of their properties, and by extension of a method due to Fréchet showed that in a complex space a pseudo-polynomial is a solution of the functional equation $\Delta^{n+1}f(x)\equiv 0$, where the increments are independent and the solution is assumed to be continuous. In this note a simpler derivation is given, depending on the results of Orliez and Mazur. Some additional properties of pseudo-polynomials are also considered. (Received May 21, 1936.)

269. Mr. Ivan Niven and Professor F. S. Nowlan: *A note on association by quartic matric polynomials.*

This paper extends results obtained by Dr. M. M. Flood, which were reported upon by him to this Society (abstract 39-3-92). Dr. Flood very kindly gave the authors access to his work. Dr. Flood's paper dealt with necessary and sufficient conditions for association by linear, quadratic, and cubic matrix polynomials. The present paper determines necessary and sufficient conditions for association by quartic matric polynomials. Unfortunately the relations obtained are extremely involved and do not suggest a solution of the general problem. (Received May 21, 1936.)

270. Mr. P. O. Bell: *Covariant configurations associated with a general curved surface at one of its points.*

In a recent paper the author completed the geometric characterization of the series of tetrahedrons associated with canonical developments for the equation of a general curved surface at one of its points. This fundamental characterization is used to determine certain configurations which are covariantly associated with the surface at one of its points. The geometry of these configurations is then studied with particular emphasis on the relations of these to previously known configurations. Among the configurations which are located and studied are cubic scrolls, twisted cubics, plane cubics, tricuspidal quartics, conics, and sequences of curves associated with a given covariant curve associated with the surface at one of its points. (Received May 21, 1936.)

271. Professor E. T. Bell: *Arithmetical consequences of a trigonometric identity.*

The formula for a product of sines leads at once to Tardy's identity (Annali di Scienze Mathematiche e Fisiche, vol. 2 (1851), pp. 287–291), and to others, from which numerous easier "Waring" problems can be seen by inspection. (Received June 9, 1936.)

272. Professor H. C. Ayres: *Existence and embedding theorems for hyperbolic systems of partial differential equations.*

A hyperbolic system of two first order partial differential equations in two independent variables and two unknown functions is studied. It is found that if such a system involves functions of class C''' a unique solution can be found which takes on prescribed initial values on any sufficiently small non-characteristic arc. Also if such a system be of class $C^{(7)}$ and a solution of class $C^{(7)}$ be given it is found that the given solution can be embedded in a one parameter family of nearby solutions. (Received June 10, 1936.)

273. Mr. Garrett Birkhoff: *The enumeration of distributive lattices.*

It is shown that the different distributive lattices of dimensions n [n finite] correspond one-one to the different partially ordered systems y of n elements. The correspondence is $Y \rightleftharpoons B^Y$, where B denotes the Boolean algebra of two elements, and the definition of "powers" directly generalizes the usual notion of powers m^n of cardinal numbers m and n. (Received May 21, 1936.)

274. Professor Salomon Bochner: *A converse of Poisson's theorem in the theory of probability.*

If each of a large number of urns contains the same total number of white and black balls, and if on drawing one ball from every urn white appears m times and black n times, then the average number of white balls in all urns taken together has a probable value that can easily be computed in terms of m/n and of the a priori probability concerning the unknown distribution of white and black in the different urns. (Received June 6, 1936.)

275. Dr. Joel Brenner: *The normal subgroups of the groups of matrices* mod p^r.

Let G be the multiplicative group of all non-singular square matrices of side n, mod p^r. This is of course the group of automorphisms of the Abelian group of order p^{nr} and type $(r, \cdots, r)$. The author has found all the normal subgroups of G, except in the case $p=2$. For a variable matrix $A=(a_{ij})$ of G, define $s(A)=\min_{ij}(a_{ij}, a_{ii}-a_{jj}, p^r)$. One can prove that $s(AB) \geqq \min(s(A), s(B)$, and $s(B\,AB)=s(A)$. The main lemma is: If a normal subgroup N of G contains any matrix A_0, then N contains all matrices $A=(a_{ij})$ with $s(A) \geqq s(A)_0$ of determinant unity mod p^r for which $a_{kk}\,1(p^{s(A_0)})$. Rules are given enumerating explicitly the normal subgroups, and stating exactly which matrices occur in each. Related results of a more miscellaneous nature are also given. (Received May 16, 1936.)

276. Professor W. C. Graustein: *Extensions of the four-vertex theorem.*

The four-vertex theorem states that an oval has at least four vertices, that is, that the curvature of an oval has at least four extrema. It is here shown that the theorem holds for larger classes of closed regular plane curves: (a) for

all simple curves; (b) for all curves with angular measure $\pm 2\pi$ or zero which have an arc of type Ω; (c) for all curves with angular measure $\pm 2n\pi$, $n \geqq 2$, which have an arc of type Ω and points of inflection. By "angular measure" is meant the algebraic angle through which the directed tangent turns when the curve is completely traced. By an "arc of type Ω" is meant a simple (open or closed) arc AB of non-negative curvature which is tangent to the same line at A and B and lies on one side of this line. (Received June 13, 1936.)

277. Dr. Marshall Hall: *An isomorphism between linear recurring sequences and algebraic rings.*

This paper makes use of a fundamental isomorphism between linear recurring sequences and algebraic rings in order to investigate the arithmetic properties of the sequences. The pattern of the periods and numerics of sequences satisfying a fixed recurrence is investigated in detail. Null sequences are considered, and necessary and sufficient conditions are found that a sequence be null for a given modulus, and also that a sequence be p-adically null.

Fundamental theorems are proved on the distribution of residues in the period, and application of these theorems to certain special recurrences yields a curious set of diophantine equations which the distribution numbers must satisfy. (Received June 15, 1936.)

278. Professor Einar Hille: *A problem in Factorisatio Numerorum.*

A study is made of the function $f(n)$ which gives the number of representations of n as a product of factors greater than one, two representations being considered identical if and only if they contain the same factors written in the same order. Estimates are given of $\sum f(n)$ if the summation is extended over all n of the form $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ where the sum of the exponents is a given integer and the primes are fixed or over all n of the given form which are less than a given N, and so on. Estimates of $f(n)$ itself are obtained as a particular case. The essential tools are functional equations, generating power series and Dirichlet series, and the Ikehara-Wiener theorem. Extensions are obtained to the case of factorizations of ideals in algebraic fields and in certain congruence fields. (Received May 22, 1936.)

279. Professor G. M. Merriman: *Concerning polynomials simultaneously orthogonal on more than one curve.*

Szegö has recently exhibited (Transactions of this Society, vol. 37 (1935), pp. 196–206) a complete list of the polynomials $p_n(z)$ in a complex variable which are simultaneously orthogonal with respect to suitable norm functions on all the level curves of a given family. In the present paper it is shown that if the same set of polynomials is simultaneously orthogonal (with respect to an arbitrary norm function) on more than one of a family of level circles or of level ellipses, then the set is orthogonal on all the level curves of the family. (Received May 27, 1936.)

280. Professor Oystein Ore: *On the theorem of Jordan-Hölder.*

This paper contains an analysis of the theorem of Jordan-Hölder for arbitary structures. The former investigations on the theorem of Jordan-Hölder in structures satisfying the Dedekind axiom (Dedekind, Garrett Birkhoff, Ore) correspond to the case of principal series and not to composition series in groups. By introducing normal elements in an arbitrary structure it is possible to derive a theorem which in many ways is analogous to the general theorem of Schreier-Zassenhaus for groups. The analogy is, however, not complete. (Received May 22, 1936.)

281. Mr. J. C. Oxtaby: *The category and Borel class of certain subsets of* L_p.

The subsets of $L_p([0, 1])$, $p \geqq 1$, corresponding to continuous functions, Riemann integrable functions, upper and lower semicontinuous functions, are shown to be $F_{\sigma\delta}$ sets of first category. The set $L_q C L_p$ for $q > p$ is shown to be F_σ and likewise of first category. The method consists in obtaining a representation for each set in terms of closed sets, utilizing boundedness properties of the functions. (Received May 28, 1936.)

282. Dr. S. Saks: *On some functionals.* II.

This paper contains some extensions of and corrections to, the results obtained in a previous paper published in the Transactions under the same title. (Received June 15, 1936.)

283. Professor M. H. Stone: *Applications of Boolean algebras to general topology.*

This paper carries the author's treatment of general topology by the use of Boolean algebras somewhat further. In addition to results previously announced in this Bulletin and in the Proceedings of the National Academy, it offers a discussion of the maps of completely regular spaces in Boolean spaces. This discussion involves the analysis of the relations between the algebraico-topological structure of the class of all bounded continuous functions on a topological space and the topological structure of the underlying space. The existence of bicompact Hausdorff extensions of a completely regular space is discussed in detail. (Received June 5, 1936.)

284. Professor J. L. Walsh and Mr. W. E. Sewell: *Note on degree of approximation to an integral by Riemann sums.*

Let $f(x)$ be defined and integrable Riemann for $a \leqq x \leqq b$. The authors investigate here the degree of approximation to the Riemann integral by the corresponding Riemann sum when equidistant ordinates are used. They consider continuous functions, functions of bounded variation, functions continuous except for finite jumps, and functions for which there are approximating polynomials converging with a given degree. A typical result is: *Let the function* $f(x)$ *be defined in the interval* $0 \leqq x \leqq 2\pi$ *and have the property that the trigonometric polynomials* $S_N(x) = \sum_{n=0}^{n=N} (a_{Nn} \cos nx + b_{Nn} \sin nx)$ *of respective orders* $N = 0, 1, 2, \cdots$, *exist such that* $|f(x) - S_N(x)| \leqq \epsilon_N$. *Then one has* $\left|\int_0^{2\pi} f(x)dx - (2\pi/N)\sum_{k=1}^{N} f(2k\pi/N)\right| \leqq 4\pi\epsilon_{N-1}$, $N > 0$. (Received May 10, 1936.)

285. Dr. Rufus Oldenburger: *Rational equivalence of a form to a sum of pth powers.* Preliminary report.

In a paper in the 1933 Proceedings of the Cambridge Philosophical Society Bronowski translated into geometry the problem of the equivalence of a p-ic to $y_1^n + \cdots + y_r^n + \lambda_1 \nu_1 + \cdots + \lambda_s \nu_s^n$. The linear transformations used by Bronowski were not non-singular, and the translated problem does not give a solution of the algebraic problem. The present paper solves the problem of the equivalence in a given field of a general p-ic form to the canonical form $ax^p + \cdots + cw^p$. The problem is treated with the aid of the author's theory of non-singular multilinear forms to appear in the forthcoming issue of the Transactions of this Society, and arithmetic invariants, which have been studied by Hitchcock and the author. The arithmetic invariant method of approach is a novel one in the theory of forms of degree higher than two. Also, the attainment of results for general p-ic forms, $p \geqq 3$, by treating these as special cases of multilinear forms is new. (Received March 6, 1936.)

286. Mr. Garrett Birkhoff: *Lie groups simply isomorphic with no linear group.*

It is shown that there exists a three-parameter Lie group, simply isomorphic even in the purely algebraic sense with no linear group. Further, every "hypercentral" Lie group of matrices is locally analytically isomorphic with a Lie group having the same defect. This contains the result announced in abstract 42-5-240. (Received June 8, 1936.)

287. Professor R. V. Churchill: *On an expansion of the inverse Laplace transformation and its application.*

The success of the Laplace transformation method of solving boundary problems in partial differential equations depends largely upon the possibility of finding the inverse transformation of the known function which appears as the solution of the transformed problem. In this paper conditions on the function are found which are sufficient to permit an expansion of its inverse transformation in a convergent series. This series is a generalization of the Heaviside partial fractions expansion. The new conditions are found with the aid of those of Tamarkin (Transactions of this Society, vol. 28 (1926), pp. 417–425) for the solution of the Laplace integral equation in the form of a complex integral. The use of the series and the conditions is illustrated by solving a problem in the forced longitudinal vibrations of a prismatic bar. (Received June 30, 1936.)

288. Professor J. L. Doob: *Stochastic processes depending on a continuously varying parameter.*

Let Ω^* be the space of all functions $x(t)$ defined for $-\infty < t < \infty$. It has been shown by Kolmogoroff that a non-negative completely additive function of sets, $P^*(\Lambda^*)$ can be defined on the Borel system of sets determined by the sets of elements $x(t)$ satisfying conditions of the form $a_j < x(t_j) < b_j$, $j = 1, \cdots, n$. A set in this Borel system, or one which differs from such a set by a subset of a

set for which P^* vanishes, will be called P^*-measurable. The exterior P^*-measure of any subset of Ω^* is defined as the greatest lower bound of the P^*-measures of the P^*-measurable sets which contain it. If Ω is a subset of Ω^* of exterior P^*-measure 1 (we are supposing that $P^*(\Omega^*)=1$) we define a measure $P(\Lambda)$ for any set Λ which is the intersection of Ω with a P^*-measurable set Λ^*: $P(\Lambda)=P(\Lambda^*)$. The conjunction of such a space Ω with the measure function $P(\Lambda)$ is defined as a stochastic process. For each value of t, $x(t)$ can be considered as a measurable function on Ω, that is, a chance variable. Stochastic processes have been previously defined as one-parameter families of chance variables. The present definition thus includes the old one, but also makes possible an examination of the characteristics of the functions $x(t)$ in t, such as continuity properties, for example. (Received June 22, 1936.)

289. Professor W. L. Duren: *A problem of Zermelo in the calculus of variations.*

The problem of minimizing the integral $\int_{x_1}^{x_2} f(x, y, y', \cdots, y^{(n)})dx$ in a class of arcs $y=y(x)$ joining the points (x_1, y_1) and (x_2, y_2) is usually treated by transforming it into a problem of Lagrange. An arc E may be said to furnish a relative minimum of order K if it minimizes the integral with respect to arcs having elements $(x, y, y', \cdots, y^{(K)})$ in a neighborhood of those of E. This paper presents sufficient conditions for relative minima of order $K(K=0, 1, \cdots, n)$ whereas the general theory of the Lagrange problem furnishes no information about relative minima of order less than $n-1$. (Received July 1, 1936.)

290. Professor Philip Franklin: *The four color problem.*

In 1920 the author gave certain reductions for the four color problem. By combining these with earlier ones due to Kempe and Birkhoff he showed that a map, not colorable in four colors, must contain at least 26 regions. In 1926 Reynolds succeeded in increasing this number to 28. His argument was simplified by the use of certain results due to Errera who in 1924 proved that every uncolorable map must contain at least 13 pentagons. In the present paper a number of new reducible configurations are treated. Some of these, involving pentagons and heptagons, were suggested by their appearance in simple maps devoid of previously known reducible configurations. By using these new reductions, the following two theorems are proved: Every uncolorable map contains at least 15 pentagons. Every uncolorable map contains at least 32 regions. (Received June 30, 1936.)

291. Dr. Aline H. Frink: *General distance functions and the metrization problem.*

In proving that under certain conditions a topological space is metrizable, a distance function is often introduced satisfying, instead of the triangle axiom, the condition: If $ab<\epsilon$, and $cb<\epsilon$, then $ac<z\epsilon$. It is shown that if this condition holds, a metric satisfying the triangle axiom can be defined directly, without making use of Chittenden's theorem which says that a space with a uniformly regular distance function is metrizable. The method also leads to a much simpler proof of Chittenden's theorem itself, since it does not depend on

the result that a normal space has non-constant continuous functions. For the case of the metrization conditions of Alexandroff and Urysohn, Niemytzki, and W. A. Wilson, it is shown that a triangle axiom metric can be introduced directly in terms of the original conditions. New metrization conditions are also given for spaces having an unsymmetric distance function. (Received June 25, 1936.)

292. Professor Orrin Frink: *Geodesic continua in abstract metric space.*

Menger has given a definition of geodesic arcs in abstract metric space, and has proved an existence theorem for them. By extending the definition to geodesic continua, and proving that the Carathéodory linear measure function is lower semicontinuous for continua, existence theorems for geodesics are obtained with very general end conditions, including conditions analogous to those of Morse in *The Calculus of Variations in the Large*. In the special case where there are just two end conditions, the geodesic continua obtained are shown to be arcs. In addition to the great generality of the end conditions and of the underlying space, the Carathéodory linear measure function used is general enough to include many of the functionals of the calculus of variations defined in terms of a single integral. (Received June 25, 1936.)

293. Dr. Max Herzberger: *A general theorem for rotationally symmetrical optical systems.*

The author demonstrates a new simple theorem, which holds true for skew rays in rotation symmetric optical instruments, and draws from it several interesting conclusions. One of them is the existence of a kind of Gaussian optics on skew rays, another is, for instance, the fact that these equations allow an integration under certain conditions, and so allow us to determine the law which must hold true in a perfect photographic or microscopic objective. (Received July 1, 1936.)

294. Professor E. V. Huntington: *Postulates for assertion, conjunction, negation, and equality.*

The purpose of this paper is to present an "abstract" mathematical theory so constructed that one of its "concrete interpretations" shall be the calculus of propositions in ordinary logic. The "base" here selected is $(K, T, \times, ', \equiv)$, where K is a class of elements $a, b, c, \cdots$ (interpretable as "propositions" or "sentences"); T is a subclass (interpretable as "asserted" propositions); ab (read a times b) is the result of a binary operation (interpretable as the "conjunction" of a and b); a' (read a dash) is the result of a unary operation (interpretable as the "contradictory" of a); and $a \equiv b$ (read a quad b) is the result of another binary operation (interpretable as the "equalization" of a and b). By definition, "$a = b$" means "$a \equiv b$ is in T"; the properties of "$=$" are not presupposed but are derived from the postulates on the selected base. The distinction between effective implication $[ab \equiv a]$ and Lewis's strict implication $[ab' \equiv Z$ (where $Z = aa')]$ is clearly analyzed; and both are contrasted with Russell's material implication $[(ab')']$. Proofs are given in full; no previous

acquaintance with Boolean algebra is required. (An abstract of part of the paper is given in the Proceedings of the National Academy of Sciences, May, 1936.) (Received June 30, 1936.)

295. Professor M. H. Ingraham and Dr. Margarete C. Wolf: *Relative linear sets and similarity of matrices whose elements belong to a division algebra.*

Given an $n \times n$ matrix M with elements in a quasi-field, a polynomial $g = \sum \lambda^i a_i$ and an $n \times 1$ vector ξ, $g(M) \cdot \xi$ is defined to be the vector $\sum M^i \xi a_i$. If the a_i's belong to a field, this is ordinary multiplication of ξ by $g(M)$. A set of vectors $\xi_1 \cdots \xi_k$ is said to be linearly independent relative to M if $\xi_i \neq 0(i)$ and if $\sum g_i(M) \cdot \xi_i = 0$ implies $g_i(M) \cdot \xi_i = 0(i)$. A set of vectors is a linear set relative to M if the sum of every two vectors is in the set and if $g(M) \cdot \xi$ is in the set for every vector ξ of the set and every polynomial g. A theory of relative linear sets is given yielding proper basis, minimal g's such that $g(M) \cdot \xi = 0$, and so on. From this is readily deduced a complete rational theory of the similarity of matrices whose elements belong to a division algebra. As a special case a simple rational treatment of the ordinary theory of similarity of matrices with elements in a field is obtained. (Received June 29, 1936.)

296. Mr. J. Q. Jordan and Dr. Walter Leighton: *On convergence of continued fractions.*

Let (1) $[1 + K_1^{\infty}(x_n/1)]$ be a continued fraction in which the x_n are arbitrary complex numbers $\neq 0$. This paper establishes the transformations on the x_n which permute the convergents of (1) in sets of three and in sets of four. These results are used to establish many new simple convergence criteria for (1) of which a characteristic example is the following. The continued fraction (1) converges if $|x_{4n+1}| \leqq \frac{1}{4}$, $|x_{4n+2}| \geqq 3$, $|x_{4n+3}| \geqq 4$, $|x_{4n+4}| \leqq \frac{1}{4}$, $(n = 0, 1, 2, \cdots)$. (Received June 24, 1936.)

297. Professor Walther Mayer and Professor T. Y. Thomas: *Analytic arcs in analytic manifolds.*

It is proved that any analytic arc $C(t)$ of an analytic manifold $\mathfrak{M}$ can be embedded in an n-parameter family of analytic arcs $C(t, a)$. This result and a lemma on the continuation of an analytic function over an open set containing an analytic arc in $\mathfrak{M}$ are used to show that the set of all points Q which can be joined to an arbitrary point P of $\mathfrak{M}$ is both open and closed. As this set is non-vacuous it follows that any two points of $\mathfrak{M}$ can be joined by an analytic arc. The methods employed are such that this result admits an immediate extension to any finite number of points of the analytic manifold. (Received June 26, 1936.)

298. Mr. John Riordan: *Recurrence relations for the moments of Bernoulli and Poisson frequency distributions.*

The mixed difference recursion formulas for moments about the origin and the mean of Bernoulli and Poisson frequency distributions, given by A. T. Craig (this Bulletin, vol. 40 (1934), pp. 262–264), are shown to have power-

series solutions, the coefficients of which satisfy partial-difference recurrence relations. For the Bernoulli moment about the origin and the Poisson moments of both kinds, the relations are relatively simple. For the Bernoulli moment about the mean, the solution seems most simply expressed in terms of coefficients satisfying a pair of partial difference equations. Triangular arrays for the coefficients are given, covering the first ten moments about the origin and the first nine about the mean. (Received June 24, 1936.)

299. Dr. J. B. Rosser: *Extensions of some theorems of Gödel and Church.*

We shall say that a formal logic is simply consistent if there is no formula A such that both A and $\sim A$ are provable. It is proved for a large class of formal logics (roughly characterized by the phrase "adequate for elementary number theory") that: (1) Simple consistency implies that the logic is not adequate to prove its own consistency. (2) Simple consistency implies the existence of a formula B such that neither B nor $\sim B$ is provable. (3) Simple consistency implies the absence of any *Entscheidungsverfahren* for the logic. (2) is an extension of a theorem of Gödel and (3) an extension of a theorem of Church in having "simple consistency" in the hypothesis instead of "ω-consistency." (1) is the well known Gödel theorem, but is proved for a larger class of logics. (Received June 18, 1936.)

300. Mr. W. E. Sewell: *A note on approximation by polynomials in the sense of least pth powers in the complex domain.*

The results of Dunham Jackson (this Bulletin, vol. 36 (1930), pp. 851–857; vol. 37 (1931), pp. 883–890) on approximation in the sense of least pth powers are sharpened in the case of a region bounded by a Jordan curve with corners, and extended to include a region bounded by an arbitrary rectifiable Jordan curve. (Received June 22, 1936.)

301. Professor T. Y. Thomas: *On simple analytic arcs.*

In this paper a number of lemmas are proved on the basis of which the following theorem is obtained: Any finite number of distinct points $P_1, \cdots, P_r$ of a connected analytic manifold of dimensionality $n \geqq 2$ can be joined by a simple analytic arc $C(t)$, where $0 \leqq t \leqq 1$, such that as t increases from 0 to 1 the arc passes through the points in the given order. (Received June 26, 1936.)

302. Professor T. Y. Thomas: *On the covering of analytic arcs by coordinate neighborhoods.*

A coordinate neighborhood of an analytic manifold $\mathfrak{M}$ is said to be proper if it is homeomorphic to the interior of an n-dimensional euclidean sphere. It is said to be improper if it is merely the continuous map of a (proper) coordinate neighborhood of the n-dimensional euclidean space. The following two theorems are proved. *First,* any analytic arc $C(t)$ where $0 \leqq t \leqq 1$ is contained in an improper coordinate neighborhood Δ in $\mathfrak{M}$, the coordinates $y^1, \cdots, y^n$ of Δ being such that $-a < y^1 < 1+a$, $|y^j| < b$ for $j = 2, \cdots, n$, where a and b are sufficiently small positive constants, and such that the arc $C(t)$ is the portion

$0 \leqq y^1 \leqq 1$ of the y^1 axis. *Second*, any simple analytic arc $C(t)$, where $0 \leqq t \leqq 1$ in an analytic manifold $\mathfrak{M}$, admits a proper coordinate neighborhood ϑ with coordinates $y^1, \cdots, y^n$ such that $-\lambda < y^1 < \lambda + 1$, $|y^j| < \lambda$ for $j = 2, \cdots, n$, where λ is a positive constant, and such that with respect to the coordinates $y^1, \cdots, y^n$ in ϑ the arc C is the portion $0 \leqq y^1 \leqq 1$ of the y^1 axis. (Received June 26, 1936.)

303. Professor T. Y. Thomas: *Fields of parallel vectors in the large.*

Let $\mathfrak{M}$ be a simply connected analytic manifold over which an affine connection Γ is defined (affinely connected space). The following theorem is proved: The space $\mathfrak{M}$ will admit K (or more) independent fields of parallel contravariant vectors with components which are analytic functions of the coordinates of the neighborhoods of $\mathfrak{M}$ if, and only if, a certain polynomial $R_K(B)$ vanishes over $\mathfrak{M}$. Here $R_K(B)$ is a polynomial in the components of the curvature tensor B and its first n covariant derivatives, where n is the dimensionality of the space. An analogous result of course holds for fields of parallel covariant vectors. (Received June 26, 1936.)

304. Professor T. Y. Thomas: *Riemann spaces of class one and their characterization.*

If necessary and sufficient conditions for the existence of a property P of a (real) space are given by conditions of the form $F_1 = 0$, $F_2 \neq 0$, $F_3 > 0$, $F_4 \geqq 0$ where the F's represent polynomials in the structural functions of the space and their derivatives to a specified order, these conditions are said to constitute an algebraic characterization of the property P. This paper deals with the algebraic characterizations of Riemann spaces as spaces of class one. In connection with the solution of this problem we have defined an integer invariant of a Riemann space which we have called the type number of the space. If the type number is one, the space is flat and hence fails to be of class one. We have excluded those Riemann spaces of type number two in as much as the discussion of such spaces requires essentially different methods than those of higher type number, and it has therefore been thought best to make these spaces the occasion of a separate investigation. For all other cases the algebraic characterizations have been constructed. (Received June 26, 1936.)

305. Professor G. T. Whyburn: *On continua of condensation.*

In this paper it is shown that a necessary and sufficient condition for a compact metric continuum M to have property N in the sense of R. L. Moore (see The Rice Institute Pamphlet, vol. 23, no. 1 (1936), p. 67, or this Bulletin, vol. 42 (1926), p. 35) is that M be locally connected and that no cyclic element of M have a continuum of condensation. (Received June 29, 1936.)

306. Professor G. T. Whyburn: *On boundaries in the plane.*

Let S be a compact locally connected plane continuum. The property of being a boundary point of a complementary domain of S is not invariant, of course, under all topological transformations of S into plane continua, whereas

the property of being a boundary point of two or more such domains is invariant under such transformations. However, it is shown in this paper that if we suppose that the boundaries of no two complementary domains of S intersect, then not only the property of being a point on the boundary of a complementary domain of S but even the property of *being the boundary* of a complementary domain of S is invariant under all topological transformations of S into plane continua. (Received June 29, 1936.)

307. Professor G. T. Whyburn: *Semi-closed sets and collections.*

A set K in a metric space S is said to be semi-closed if each component of K is closed and every convergent sequence of components of K whose limit intersects $S-K$ converges to a single point. Similarly, a collection G of disjoint sets is semi-closed if each set is closed and every convergent sequence of the sets whose limit intersects $S-G$ converges to a single point. In addition to proving elementary properties of such sets and collections relating them to upper semi-continuous collections, it is shown that if F is a semi-closed subset of a compact locally connected continuum S such that no component of F separates S locally and for any irreducible cutting K of S locally and for any irreducible cutting K of S between two points $K \cdot F$ is contained either in a countable number of components of K or in a countable number of components of F, then $S-F$ is connected and locally connected. Various applications of this theorem to particular cases, such as that in which S is unicoherent or is a sphere and F is the sum of the boundaries (assumed disjoint) of the complementary domains of a continuum in S, are given. (Received June 29, 1936.)

308. Professor R. L. Wilder: *Locally connected subsets of euclidean n-space.*

This paper is an investigation of the external topological properties of closed subsets J^k of euclidean n-space $E_n (n>2)$ with a view to generalizing the work of Schoenflies on continuous curves in the plane. (A set J^k is a locally compact metric space that is locally i-connected for $i \leqq k$.) At all points a, the local Betti numbers $p^i(a, J^k)$ of a J^k are $\leqq \omega$ for $i \leqq k$, and if in addition it is semi-$(k+1)$-connected, then $p^{k+1}(a, M) \leqq \omega$. If a semi-$(n-2)$-connected J^{n-3} is embedded in E_n, then it is regularly s-accessible for $s>0$; and if it is a J^{n-2} and D is a domain complementary to it, the boundary of D is regularly 0-accessible from D. A compact continuum in E_n is a J^0 if the diameters of its complementary domains form a null sequence and the boundaries of its complementary domains are regularly s-accessible ($s \leqq n-2$) from these domains. Characterizations are obtained, in terms of external topological properties, for sets J^k for various values of k. In case $k=0$ a characterization of the Jordan continuum in E_n is obtained in terms of the properties of the bounding $(n-2)$-cycles of the complement. (Received June 26, 1936.)

309. Professor R. M. Winger: *Non-trochoidal rational curves of order n which admit dihedral groups of order* $2n-4$.

In a former paper (American Mathematical Monthly, vol. 39 (1932), pp. 578–589) projective trochoids of order n invariant under dihedral collineation groups of orders $2n-2$ and $2n-4$ were discussed. Such curves are necessarily of even order. For all values of n ($n>3$), even or odd, however, there is a one-parameter family of non-trochoidal curves invariant under a dihedral collineation group of order $2n-4$. These form the subject of the present study. The curves in question divide broadly into two classes, according as n is odd or even, corresponding to the two main classes of dihedral groups. The family includes as special cases many remarkable individuals, among which may be mentioned the self-dual curves of Wear (American Journal of Mathematics, vol. 51 (1929), pp. 482–490), which are autopolar with respect to the maximum number of conics; and certain of the polar tangent curves treated by Stratton (Dissertation, University of Washington). In a metrical setting the curves are symmetric with respect to $n-2$ axes, equispaced about a point which is also a center of symmetry when n is even. In this form the self-dual curves become the symmetric curves studied by Duncan (for n odd) (this Bulletin, vol. 40 (1934), pp. 344–352). (Received June 29, 1936.)

310. Mr. C. B. Allendoerfer: *Einstein spaces of class one.*

This paper obtains necessary and sufficient conditions that an n-dimensional Einstein space ($n>3$) be of class one; that is, that it be not flat and that it admit an imbedding in an $(n+1)$-dimensional flat space. The conditions obtained consist of (1) a set of tensor equations involving only the components of the metric tensor and their derivatives; (2) the requirement that a certain matrix be semi-definite of rank one, the elements of which are polynomials in the components of the metric tensor and their derivatives; and (3) the requirement that the mean curvature of the space be different from zero. All results are proved for a simply connected neighborhood which is covered by a single coordinate system. (Received July 2, 1936.)

311. Professor C. C. Camp: *Expansions involving a system of differential equations in which the coefficients of the parameters change sign.*

For the single equation $X'+\lambda a(x)X=0$ and the boundary conditions $X(-1)=X(1)=0$ one treats by means of the lemma:

$\lim_{|z|\to\infty}\int_c [e^{2hz}dz/2\pi iz(e^{2z}-1)=\lim_{|z|\to\infty}\int_c [e^{-2hz}dz/2\pi iz(e^{-2z}-1)]=h-\frac{1}{2}$

when $2h$ is an integer, otherwise zero, the case in which $a(x)$ is real, integrable, and has a countable number of zeros but $\int_{-1}^{1}a(x)dx=2A_1\neq 0$. The points at which $a(x)=0$ are not significant but if $\int_{-1}^{\xi_i}a(x)dx=k_iA_1$, $k_i=0, \pm 1, \pm 2, \cdots$, then the expansion of $f(x)$, consisting of a finite number of pieces, each real, continuous and possessing a continuous derivative, will not be affected unless f is discontinuous at one of the points ξ_i. In such a case the expansion will converge to the usual mean value plus terms each consisting of a linear combination of the k's multiplied by $f(\xi_i+0)-f(\xi_i-0)$. This reasoning is extended to a system of p linear equations in p parameters. (Received July 2, 1936.)

312. Professor Leonard Carlitz: *On generalized Gauss sums.*

Let $\nu(A, M)$ denote the coefficient of x^{k-1} in the reduced form of $A \pmod M$, where $A, B, \cdots, M$ are polynomials in an indeterminate x with coefficients taken (mod m), $m>0$, and M of degree k. Then the sums in question are of the form $\sum_A e(A^2H, M)$, summed over all $A \pmod M$, where $e(A, M) = \exp\{2\pi i\nu(A, M)/m\}$. These sums are evaluated by a very elementary method. Applications are made to a generalized theorem of quadratic reciprocity. (Received July 2, 1936.)

313. Professor Leonard Carlitz: *On the "singular series" for sums of squares of polynomials.*

By using the generalized Gauss sums described in the previous abstract, the "singular series" for the number of representations of a polynomial as a sum of squares is derived in an elementary way. The construction holds for the more general problem of determining the number of solutions of $M = A_1X_1^2 + \cdots + A_sX_s^2$, where M, A_j are assigned polynomials in a single indeterminate with coefficients in a finite field. Certain related problems may be treated by the same methods. (Received July 2, 1936.)

314. Professor L. W. Griffiths: *Representation by generalized polygonal numbers.*

In my paper in the American Journal of Mathematics, vol. 55 (1933), pp. 102–110, there is a summary of the problem of representation of positive integers by generalized polygonal numbers as a generalization of representation by squares. Necessary conditions that the integers $0, 1, \cdots, 34m-16$ should be represented are obtained. Here m is the positive integer defining the generalized polygonal numbers. In this paper, if the coefficients satisfy these necessary conditions, there is exhibited a positive integer M, depending only on m and the coefficients, such that every integer greater than M is represented. (Received July 3, 1936.)

315. Professor T. R. Hollcroft: *The binet of quadrics in S_3.*

Linear systems of quadrics in S_3 of dimension 1, 2, and 3 (pencils, nets and webs, respectively) have been treated extensively in the last half century, but nothing has been done specifically for a dimension greater than three. In the present paper, a linear $\infty 4$ system of quadrics is investigated. The name "binet" is suggested for such a system because and only because the dimension of the system is twice that of a net. In no other sense whatever is the binet a "double net." The properties of the binet are derived by considering the (1, 1) correspondence between the quadrics of the binet and the primes of S_4 and by studying the properties of the branch-point primal associated with this correspondence. The jacobians of the $\infty 4$ webs of the binet form another binet with a basis curve ϕ of order 10 and genus 11. The locus of axes of composite quadrics of the binet is a ruled surface of order 10 and index of irregularity 6 containing ϕ as a triple curve. The loci of contacts of quadrics of the binet are completely characterized. (Received July 2, 1936.)

316. Mr. H. M. MacNeille: *Extension of a multiplicative system to a Boolean ring.*

Consider a hypercomplex system, H, of unlimited rank, generated by an unrestricted set, K, of basis elements and an arbitrary ring, R, and such that each element of H has but a finite number of non-zero components. If K is an arbitrary multiplicative system (defined by the author, Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 47) and R is the ring of integers modulo 2, then H is a Boolean ring (defined by J. von Neumann and M. H. Stone, Fundamenta Mathematica, vol. 25 (1935), p. 353). If we suitably define an ideal, M, the quotient ring, $L=H/M$, is the smallest Boolean ring in which K can be imbedded. If K has a greatest element, then L is a Boolean algebra and addition is the Boolean symmetric difference. The author has previously obtained essentially equivalent results by another method (loc. cit., p. 49). The present method is the more elegant and the more expedient when working with the symmetric difference (as Stone does, Transactions of this Society, to appear shortly). The former method is superior when working with the ordering relation and is necessary for the consideration of infinite operations. (Received July 2, 1936.)

317. Professor A. W. Tucker: *Branched and folded coverings.*

A *branched* covering of one n-complex by another is a generalization of the covering of a sphere by a Riemann surface; *folding* is an added complication quite well described by the intuitive meaning of the world. Through rudimentary combinatorial methods two formulas are developed which relate the Euler characteristics of the two complexes concerned in the covering with those of the subcomplexes about which branching and folding occur. One formula takes account of orientation; the other does not. (Received July 3, 1936.)