to whether the book should be destined for the beginner or whether its purpose should rather be a survey of a field for experts and fellow-contributors. A little contrast can be doubtless observed between the broad treatment of some comparatively elementary situations and the preference of special and individual topics to some others of importance. But at all events the book will be useful from both points of view and it is a significant enrichment of the series of the Colloquium Lectures.

GABRIEL SZEGÖ

MORSE ON CALCULUS OF VARIATIONS

The Calculus of Variations in the Large. By Marston Morse. Colloquium Publications of the American Mathematical Society, vol. 18. New York, 1934. ix+368 pp.

The background for the theory elaborated in this volume lies in two rather distinct fields of mathematics. We have on the one hand the theory of critical points of functions of n real variables, largely created and developed by the author and his students; on the other hand, the classical calculus of variations and its modern treatment as a part of the functional calculus, to which Hadamard and Tonelli have made the fundamental contributions. The origin of the theory of critical points is perhaps to be found in the minimax principle introduced by Birkhoff in his paper on *Dynamical systems with two degrees of freedom* (see Transactions of this Society, vol. 18 (1917), p. 240). While at this early stage the connection of the theory with the calculus of variations was made clear, later developments have made the relations between the two fields much more intimate.

The calculus of variations in the large derives its interest not only from its use of functional calculus methods. More significant is the fact that it undertakes a study of the configurations to which a calculus of variations problem gives rise, namely, the extremals, with respect to important properties apart from the question whether or not they furnish extreme values for a definite integral. In the classical calculus of variations, only such extremal arcs AB were considered which contained no point A' conjugate to A. The theory with which this book is concerned, and also the earlier investigations of Birkhoff, reveal the importance of extremal arcs AB upon which there may be one or more points conjugate to A; not only arcs in the small, restricted by the exclusion of conjugate points, but also arcs in the large have to be considered for a full understanding of this theory.

A critical point x_i^0 of $f(x_1, \cdots, x_m)$ is a point at which all the first order partial derivatives of f vanish. Each critical point has a type number equal to the number of negative terms in the quadratic form which represents the second order terms of $f(x_i)-f(x_i^0)$, when reduced to a sum of squares. The author's earlier papers establish a beautifully simple set of relations between the numbers of critical points of various types of $f(x_1, \cdots, x_m)$ and the connectivity numbers of the domain R over which the function ranges. Thus the topological character of the domain of the independent variable is linked up with the critical values of the function. In the present volume these relations are extended to a functional domain. Instead of a function $f(x_1, \cdots, x_m)$, we consider now a functional $J=\int f(x, y_i, y_i')dx$, depending upon a curve $y_i(x)$, $(i=1, \cdots, m, a\leqq x\leqq b)$, that is, upon a point in a function space. In place of the n-dimensional region R of euclidean space we have a domain in function space. The critical points are now the extremals of the functional, that is, the solutions of class C'' of the Euler equations connected with this functional. It becomes necessary to set up for domains in function space topological invariants in terms of which their connectivities can be defined. We shall now pass on to a more detailed consideration of the separate chapters.

Chapter 1 (pp. 1–17) first develops the classical theory of the integral $\int f(x, y_i, y_i')dx$, for fixed endpoints, not including the theory of the second variation. The conjugate point theory is developed in considerable detail in accordance with the important role it plays in later parts of the book. Necessary conditions for a weak minimum are derived, and sufficient conditions for a strong minimum. A neat proof of DuBois-Reymond's lemma appears on page 3.

Chapter 2 (pp. 18–36) takes up the problem with variable endpoints in the form used in several of the author's papers: to minimize

$$\theta(\alpha)+\int_{x^1(\alpha)}^{x^2(\alpha)} f(x, y_i, y_i')dx.$$

The discussion of the second variation introduces us to the author's methods. The "accessory boundary problem" associated with the second variation leads to the characteristic roots, to the index of an extremal, and to the formulation of sufficient conditions in terms of characteristic roots.

In Chapter 3 (pp. 37–79) we come to the study of extremal arcs which are not restricted to be free from conjugate points. The connection between conjugate points and characteristic roots is developed further and linked up with the important concept of the index form. This is accomplished by means of the method of broken extremals. A set of points $A_1, \cdots, A_p$ is taken on the extremal g, determined by $y_i=\bar{y}_i(x)$; through each point A_i of this set a manifold is constructed, on which a point P_i is then taken. If the points P_i are sufficiently near the points A_i, two successive ones can be joined by extremal arcs of the functional

$$J^\lambda=\theta(\alpha)+\int_{x^1(\alpha)}^{x^2(\alpha)} [f(x, y_i, y_i')-(\lambda/2)\sum(y_i-\bar{y}_i(x))^2]dx.$$

The value of this functional taken along the "broken extremal," that is obtained in this way, gives rise to a function of the coordinates of the points P_i. The second order terms in the expansion of this function yield a quadratic form with coefficients depending on λ, called the index form of g. Its properties play an important part in later work. Necessary conditions and sufficient conditions for a minimum are given in terms of the index form for the problems with one and with two variable endpoints, and for periodic extremals. The index of a periodic extremal is evaluated in terms of conjugate points and of a numerical invariant, called the "order of concavity"; a necessary condition that a periodic extremal give a strong minimum is that the order of concavity be zero.

The work of the preceding chapters has shown the importance of self-adjoint boundary problems for the calculus of variations; Chapter 4 (pp. 80–106) is devoted to a systematic exposition of their theory. First comes an invariant formulation of self-adjoint boundary conditions for self-adjoint systems of differential equations of the second order in n variables. Then follows a discussion of boundary problems which involve a parameter, characteristic roots and solutions, the comparison of two problems which differ only in the boundary conditions, a general oscillation theorem, comparison of boundary problems involving distinct differential forms, and related topics. The chapter contains a large number of interesting and important theorems.

Chapter 5 (pp. 107–141) brings a study in tensor form of the parametric problem on a Riemannian manifold whose points and neighborhoods are 1–1 images of an m-dimensional simplicial circuit, a neighborhood of each point being "homeomorphic with a neighborhood of a point (x) in a euclidean m-space of coordinates" $x^1, \cdots, x^m$. For each neighborhood there is "at least one such representation" in which there is defined a positive definite form such as $g_{ij}(x)dx^i\,dx^j$, "defining a metric for the neighborhood". In a preliminary section there is a proof of the theorem on "normal coordinates" which justifies the representation of a "simple regular arc of class C^4" and a neighborhood of such an arc by a segment of a straight line in a euclidean space and a neighborhood of this segment. The theory, as developed in the earlier chapters for the non-parametric case, is here presented in an elegant and concise manner for the parametric problem $J=\int f(x, x_i)dt$. The theory of the index form and its connection with conjugate points and focal points is carried over to this problem by the use of the "normal coordinates".

The chapters whose contents have been briefly sketched thus far have introduced us to the new concepts which are more fully developed in the remaining chapters. Chapter 6 (pp. 142–192) deals with the theory of critical points of a function f on a Riemannian manifold R of the character introduced in the preceding chapter, so that in every neighborhood it can be represented as a function x of class C_2 of local coordinates $x_1, \cdots, x_n$. A point at which the first partial derivatives of x vanish is a critical point; the critical point is degenerate or non-degenerate according as the Hessian of x does or does not vanish there. If all the critical points are non-degenerate, they are isolated and hence finite in number; the function f is then called non-degenerate. In various papers the author has established the fundamental relations

$$\sum_{i=0}^{k}(-1)^{i+k}M_i \geqq \sum_{i=0}^{k}(-1)^{i+k}R_i, \qquad (k=0, 1, \cdots, n-1),$$

and

$$\sum_{i=0}^{n}(-1)^{i}M_i = \sum_{i=0}^{n}(-1)^{i}R_i,$$

which hold between the numbers M_i of critical points of type i of a non-degenerate function f and the connectivities R_i of R. The essentially new problem of the present chapter is that of the extension of these relations to the case in which degenerate critical points occur. It requires a study of critical sets σ, that is, sets of critical points at which f takes on a fixed value c. Corresponding to each such set, a set of type numbers m_k is introduced in terms of "new k-cycles" and of "newly-bounding k-cycles" and the critical set σ is said to be "equivalent" to m_k non-degenerate critical points of type k, for $k=0, \cdots, n$. The numbers m_k are completely determined by the definition of f in an arbitrarily small neighborhood of the set σ. The equivalence definition maintains the fundamental relations in the case of degenerate critical points. If f is analytic and is continued by a function Φ of the form $F(x, \mu)$ in the local coordinates on R and a parameter (μ), analytic and non-degenerate, reducing to f for $(\mu)=(0)$, then Φ will have at least m_k non-degenerate critical points of type k, near the critical set σ, provided (μ) is sufficiently near (0). It is with the proofs of these various statements that the chapter is chiefly concerned. They involve detailed topological considerations not amenable to a condensed form of statement. The chapter ends with applications of the theory to theorems on the number of normals from a point O to a manifold R in euclidean space and on the number of chords of R which are normal to R at both ends.

The methods developed in Chapter 6 are carried over in the next chapter (pp. 193–249) to the functionals of the calculus of variations discussed in Chapter 5. The independent variable now ranges over the domain Ω of curves γ, which are continuous images on R of the segment $0 \leqq t \leqq 1$, are of class D' on R, and have endpoints belonging to a terminal domain Z. The various topological concepts of the preceding chapter are defined in Ω. Between two points A^1 and A^2 of R, a set of points $P^1, \cdots, P^p$ is taken such that any two successive points of the set $A^1, P^1, \cdots, P^p, A^2$, denoted by π, can be joined by an "elementary extremal," that is, an extremal segment for which the integral J is less than a prescribed amount. The value $J(\pi)$ of the integral J taken along the broken extremal $g(\pi)$ constructed in this way is "a function ϕ of the parameters (α) locally representing its vertices A^1, A^2 and of the successive sets of coordinates (x) locally representing its vertices P^i". The critical points and critical sets of ϕ form the basis of the discussion and the connecting link between the earlier theory of critical points and the calculus of variations. A necessary and sufficient condition that a set (π), of which no two consecutive vertices coincide, be a critical point is that $g(\pi)$ be a "critical extremal". The topology of the critical extremals is investigated, resulting in theorems concerning the connectivities of Ω and of its subdomains determined by inequalities of the form $J(\pi)<b$. Deformations on Ω and a J-distance between two curves of class D' play an important part in these investigations. The details are too numerous to make a brief account feasible. They are concerned with isolated critical extremals and with critical sets of extremals. For a critical set, type numbers m_k are defined and it is shown that the sums N_k of the kth type numbers of all critical sets and the connectivities P_k of the domain Ω are connected by the inequalities $N_k \geqq P_k$. If N_i $(i=0, 1, \cdots, r)$, are finite, the inequalities of Chapter 6 are shown to hold for $k=0, 1, \cdots, r$. For a non-degenerate critical

extremal of index k, all the type numbers vanish except the kth, which is equal to 1. It follows that in the non-degenerate case "the number N_k of distinct extremals of index k" and the connectivities P_k of Ω satisfy the relations $N_k \geqq P_k$. After a further analysis of the non-degenerate problem, the problem with fixed endpoints and that with one variable endpoint are taken up. Application of the general theorems leads to conclusions concerning the number of conjugate points on extremals joining two points (Theorem 13.3, p. 239) and to certain corresponding results for focal points (Theorem 14.4, p. 244). The chapter closes with a discussion of the connectivities of the functional domain Ω (A^1, A^2) determined by the fixed-points A^1 and A^2 on an m-sphere. From the known character of the geodesics on a sphere it is easy to derive, by use of general theorems established in this chapter, the fact that the connectivities P_k of this domain are all 0 except those for which $k=p(m-1)$, $(p=0, 1, \cdots)$; for these values of k, $P_k=1$.

A new topological approach is required for the problem of closed extremals treated in Chapter 7 (pp. 250–306). The closed broken extremal with arbitrarily many vertices, taken in circular order, is the basal element; the set of all spaces representing such sets of vertices is the domain Ω whose connectivities are to be used. The central problem consists in the definition of chains, cycles, homologies, and connectivities in this space. This is done in first instance by means of a metric dependent on "elementary extremals". Then, by means of an abstract formulation of a "metric M_0 with elementary arcs" it is shown that the connectivities as defined for the space Ω are independent of the particular metric used. In addition to the usual postulates for a metric space S constituted by points $P, Q, R, \cdots$, two further postulates are used. After the uniform continuity of the distance PQ on S has been established and a continuous map of one such space on another has been defined, the term "simple arc γ" on S is introduced to designate the homeomorph of the segment $0 \leqq t \leqq 1$. The additional postulates are to the effect that for every space S there shall exist a positive constant ρ, so that for every pair of points P and R for which $0 < PR \leqq \rho$ there shall exist one simple arc $[PR]$ of such nature that for Q on $[PR]$, and between P and R, $PR = PQ + QR$, while for any point Q not on R, the inequality $PR < PQ + QR$ shall hold. The topological machinery that is obtained in this way is adequate to the needs of the problem. The connectivities in this space of sets of points in circular order are called circular connectivities. Theorems analogous to those obtained in the previous chapter are now proved, thus establishing for closed extremals fundamental relations between the circular connectivities of Ω and the index numbers of the extremals.

The final chapter (pp. 307–358) is concerned with a generalization of Poincaré's theorem concerning the existence of closed geodesics on a convex surface belonging to a family S_α of surfaces, containing an ellipsoid of unequal axes for $\alpha=0$ and depending analytically on α. The author asks for "numerical characteristics of sets of closed geodesics on $R \cdots$ sufficient to guarantee the existence of a corresponding set" of closed geodesics on any other admissible Riemannian manifold homeomorphic with R. He proves that these sets of geodesics vary analytically with respect to any parameter with respect to which the manifolds vary analytically. The determination of these characteristics for geodesics on the m-sphere furnishes criteria for the existence of closed extremals corresponding to a large class of functionals defined on manifolds homeomorphic to a sphere. The method used here is contrasted with Birkhoff's method of fixed points of transformations. The circular connectivities introduced in the preceding chapter are now determined for the m-sphere by means of an analysis of the closed geodesics on ellipsoids. The analysis is full of interesting details, it utilizes the topological concepts defined in earlier parts of the book, and leads to the following simple result. The kth circular connectivity of the m-sphere is the number of distinct integral solutions, i, j, r of the diophantine equation $k=m+i+j-4+2(r-1)(m-1)$ under the restrictions $m+1\geqq i>j>0$, $r>0$. This result leads to existence theorems for closed extremals, of which the following is an example (Theorem 9.2, page 352): "Let R be a Riemannian manifold homeomorphic with an m-ellipsoid $E_m(a)$ for which $a_1> \cdots >a_{m+1}$. If the constants (a) are sufficiently near unity, there exists a set G of closed extremals which is topologically related on R to the principal ellipses on $E_m(a)$, and which has a kth type number sum at least as great as the number of principal ellipses of $E_m(a)$ of index k." At the end of the chapter come two continuation theorems on geodesics the first of which may roughly be stated as follows: If R_α is a 1-parameter family of Riemannian manifolds and K a critical set of closed geodesics on $R_{\alpha 0}$, then, if $|\alpha-\alpha_0|$ is sufficiently small and not zero, the manifold R_α contains a finite ensemble K_α of critical sets of closed extremals which "tends to" a subset of K as α tends to α_0 and for which the sum of the kth type numbers is not less than for K.

The account which has been given of the contents of this book is of necessity very sketchy, particularly as far as the last two chapters are concerned; and no mention has been made of work which has appeared since the publication of this volume. This inadequacy is not wholly due to limitations of space, but it is to a large extent the result of the reviewer's limitations; the book makes considerable demands upon the reader. But it is a fascinating and stimulating book. It is to be expected that its reading will lead many mathematicians to engage in the study of the problems which are suggested; the extensive bibliography will be helpful in initiating him in the literature.

The book represents a signal contribution to mathematics. The author is to be heartily congratulated on its completion and the American Mathematical Society deserves our thanks for having made its publication possible.

Very few misprints have come to the reviewer's notice, none of serious character. One desideratum has to be mentioned: the use of the book would be greatly facilitated by an index of the large number of special terms which are used and which are of such character as not to be readily retained by a reader who is not himself working in the field. It is to be hoped and expected that a second edition of the book will soon furnish the opportunity for supplying this need.

ARNOLD DRESDEN