ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

391. Professor E. P. Lane and Professor M. L. MacQueen: *Asymptotic curves on a surface.*

The purpose of this paper is to make some contributions to the theory of the asymptotic curves on an analytic non-ruled surface in ordinary projective space. The coordinates of a variable point on the surface are regarded as solutions of a system of linear homogeneous partial differential equations which are written in Fubini's canonical form. Power series expansions in non-homogeneous projective coordinates for the parametric asymptotic curves on the surface are computed to terms of higher degree than have hitherto been considered. By means of these power series a study is made of the asymptotic curves regarded as space curves. Some progress is made in this paper toward the solution of the problem of finding all surfaces on which the asymptotic curves are twisted cubics. Some properties of such surfaces are presented. (Received September 1, 1936.)

392. Professor L. M. Blumenthal: *A note on horn angles and abstract metrics.*

Kasner has shown that attached to an ordered pair C_1, C_2 of analytic curves passing through a point P in the same direction, is the unique (regular) conformal invariant $I_{12} \equiv (d\gamma_2/ds_2 - d\gamma_1/ds_1)/(\gamma_2-\gamma_1)^2$, $(\gamma_1 \neq \gamma_2)$; the γ's are the curvatures of C_1, and C_2, respectively, at P. A recent paper by Kasner and Comenetz (Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 303–309) derives "a fundamental identity in conformal plane geometry, relating the conformal invariants of adjacent horn angles." An inequality satisfied by the three conformal invariants of three horn angles formed by three curves at a point is also obtained. In this note an abstract space K is considered whose points are ordered pairs of real numbers, with the distance P_1P_2 of the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ defined as $P_1P_2 = (y_2-y_1)/(x_2-x_1)^2$, $x_2 \neq x_1$. The group having the distance defined above as the unique invariant of two points is $x' = ax+b$, $y' = a^2y+c$. The fundamental identity found by Kasner and Comenetz is merely the four-points relation for the space K, and is easily exhibited in determinantal form, while the triangle inequality for three horn angles is, in this interpretation, immediate. Many properties of the space K

give rise to new problems in the conformal geometry of horn angles. (Received September 24, 1936.)

393. Professor L. M. Blumenthal: *New metric foundations of hyperbolic geometry.*

New necessary and sufficient conditions (in terms of distance relations of certain *quadruples* of points) are found that to each element of a given semi-metric space M an ordered set of n real numbers may be attached, in such a way that if x and y are any two elements of M, and $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ are the n-tuples attached to x and y respectively, then the hyperbolic distance function applied to the two n-tuples shall determine a number equal to the previously defined distance xy. (Received October 8, 1936.)

394. Professor T. R. Hollcroft: *The web of quadrics.*

The characteristics of a web of quadrics, with and without basis elements, have never been completely or accurately given. This paper gives a brief summary of the results obtained by many mathematicians during the past seventy-five years, correcting errors and completing the sets of characteristics of both the web of quadrics and its associated involution. The characteristics of a web of quadric cones (a web of quadrics with a basis double point) are derived. The nature of these characteristics is entirely different from that of a web of non-singular quadrics. (Received September 1, 1936.)

395. Dr. Reinhold Baer: *Primary abelian groups and their automorphisms.*

To every group A of (proper) automorphisms of the primary abelian group G corresponds the subgroup $F(A)$ of its fixed elements in G and the subgroup $C(A)$, generated by the commutators $x-xf$ with x in G and f in A. Those subgroups of G which are groups $F(A)$ and those which are groups $C(A)$ for a suitable group A of automorphisms of G are characterized by their structure and by their situation in G. The centralizer of the group of all the automorphisms of G appears to be essentially the group of those automorphisms of G which map every subgroup of G upon itself and this allows one to characterize those primary abelian groups whose group of automorphisms is abelian. (Received September 23, 1936.)

396. Dr. Reinhold Baer: *Dualism in abelian groups.*

A dualism d in an abelian group G is a one-one correspondence between the subgroups of G such that $S \leqq T$ if, and only if, $Td \leqq Sd$ and such that S and $G/(Sd)$ are isomorphic. The existence of a dualism in a finite abelian group has been noticed by E. Steinitz. It is remarkable that some sort of converse holds true. For there exists a dualism in the abelian group G if, and only if, G is a group without elements of infinite order whose primary components are finite. If there exists a dualism, then there exists even a dualism d which satisfies: A/S and Sd are isomorphic; there exists an automorphism of G, mapping S upon T, if and only if there exists an automorphism of G, mapping Sd upon Td. (Received September 10, 1936.)

397. Dr. Reinhold Baer: *Equivalence of algebraic extensions.*

If A and B are equivalent algebraic extensions of their common subfield C, then the same equations in C have solutions in A and in B. The converse holds true if, for example, A is separable with regard to C. If A and B are not separable with regard to C, then it is possible to prove an analogous proposition under somewhat weaker assumptions. (Received September 10, 1936.)

398. Dr. George Comenetz: *Conformal geometry on a surface.*

This paper generalizes to the case of curved surfaces certain theorems on plane conformal geometry due to Kasner (Proceedings of the Fifth International Congress of Mathematicians, Cambridge, vol. 2 (1912), p. 81; Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 303; and unpublished work). In this paper are considered properties of figures on a surface which remain invariant when the surface is transformed conformally into any other surface. Let two curves be tangent at a point O, and let Γ, G be their respective geodesic curvatures at O. Let Γ_1, G_1 denote the values of the first derivatives of curvature with respect to arc-length at O; Γ_2, G_2 the second derivatives, $\cdots$. Then if the curves have first order contact, the quantity $(G_1-\Gamma_1)/(G-\Gamma)^2$ is an invariant, just as in the plane case. But for second order contact $(G=\Gamma)$, the invariant found is $[4(G_1-\Gamma_1)(G_3-\Gamma_3)-5(G_2-\Gamma_2)^2-4(\Gamma^2+K)(G_1-\Gamma_1)^2]/[G_1-\Gamma_1]^3$, where K is the Gaussian curvature of the surface at O. For a curvilinear right angle, the new invariant differs from the known invariant in the plane case by the addition of $-K(G_1-\Gamma_1)+2(GK_s-\Gamma K_\sigma)+(-gK_s+\gamma K_\sigma+K_{s\sigma}+K_{\sigma s})/2$ to the numerator, where the subscripts σ, s indicate differentiation with respect to arc-length along the curves of an arbitrary orthogonal net tangent to the right angle at O, and γ, g are the curvatures of the curves of the net at O. (Received September 19, 1936.)

399. Dr. George Comenetz: *Schwarzian reflection on a surface.*

Let C be a given curve on a surface S. The Schwarzian reflection, or *conformal symmetry*, determined by C may be defined as the anti-conformal transformation of S into itself which leaves the points of C fixed. Take any curve D on S intersecting C at a point O, and let $\overline{D}$ be the image of D under the symmetry. Let γ be the geodesic curvature of C at O; γ_1 the first derivative of curvature with respect to arc-length, at O; γ_2 the second derivative; $\cdots$. Let Γ, Γ_1, Γ_2, $\cdots$, be the corresponding quantities for D at O; and $\overline{\Gamma}$, $\overline{\Gamma}_1$, $\overline{\Gamma}_2$, $\cdots$, for $\overline{D}$ at O. Call the angle from C to D at O, θ. Then $\Gamma+\overline{\Gamma}=2\gamma\cos\theta$, and $\Gamma_1+\overline{\Gamma}_1=2\gamma_1\cos 2\theta$, just as when S is a plane. But $\Gamma_2+\overline{\Gamma}_2=2\gamma_2\cos 3\theta-2\gamma_1(\Gamma-\overline{\Gamma})\sin 2\theta+2\gamma(\Gamma_1-\overline{\Gamma}_1)\sin\theta+K_s\sin\theta\sin 2\theta$, where K_s is the derivative at O of the Gaussian curvature of S, with respect to arc-length perpendicular to C. The expression for $\Gamma_3+\overline{\Gamma}_3$ is obtained when $\theta=0°$ and $\theta=90°$, and is applied to calculate the conformal invariant of a right angle from the invariant of a horn angle of second order contact. When S is a plane, the results become those obtained by Kasner (references in previous abstract). (Received September 19, 1936.)

400. Dr. A. D. Fialkow (National Research Fellow): *Initial motion at a point of equilibrium.*

A plane positional field of force gives rise to the three-parameter family of dynamical trajectories. A differential-geometric characterization of this system of curves at regular points of the field has been given by Kasner. The author considers similar questions in the neighborhood of an isolated point of equilibrium. The following theorem is analogous to the first of Kasner's set of characteristic properties: The cubics osculating each of the ∞^1 trajectories passing through a point of equilibrium in the same direction at the given point have asymptotes which form a pencil whose center lies on the initial tangent. The qualitative nature of the lines of force near the point of equilibrium has been studied by Poincaré, Bendixson, and others. By means of some of their results, a few elementary connections between the differential geometry of the paths and the general character of the lines of force are found. (September 24, 1936.)

401. Dr. R. A. Harrison: *Cremona webs of surfaces in S_3 without base curves.*

Cremona transformations $T_{n-n'}$, in three-way space S_3, in which n' has the maximum value n^2, have no fundamental curves for the web of surfaces (ϕ_n) of order n. The surfaces of the web have a common $(n-1)$-fold point; and, if the tangent cone at this point, the vertex, is not fixed, the transformations are necessarily of order 2, 3, or 4. After consideration of these transformations, the more general case, in which the tangent cone at the vertex is fixed for the system, is taken up. Finally, T_{2-4} is extended to T_{2-8} in S_4 and generalized to $T_{2-2(n-1)}$ in S_n, while the possible generalization of T_{3-9} and T_{4-16} is noted. (Received October 5, 1936.)

402. Professor Edward Kasner: *Conformal theorems on triplets of curves.*

A triplet T is defined as three curves $C_1C_2C_3$ passing through a common point in a common direction. The three horn angles determine three conformal invariants I_{12}, I_{23}, I_{13}, or three natural measures M_{12}, M_{23}, M_{13} (see Proceedings of the National Academy of Sciences, April, 1936), for which a new type of triangular inequality is valid. In the equality case $M_{12}+M_{23}=M_{13}$ the triplet is called wide-open (or linear). For the general triplet C_3 may be replaced by C_3' in such a way that M_{12} and M_{23} are the same and the new triplet $C_1C_2C_3'$ is linear. Two triplets whose three corresponding angles have equal measures are not necessarily conformally equivalent. Thus $C_1C_2C_3$ may be replaced by $C_1C_2C_3{}^*$. The curves C_3 and $C_3{}^*$ are then said to be "linear symmetric" with respect to the linear system determined by C_1C_2. If the three horn angles in T are bisected (by Schwarzian or conformal symmetry), the new triplet has measures half of the old. When bisection is iterated indefinitely a limiting curve C_0 is obtained which may be said, in a new conformal sense, to be the trisection point of the three median linear systems. (Received September 26, 1936.)

403. Mr. A. P. Morse: *Convergence in variation and related topics.*

In view of a result due to Adams and Clarkson (this Bulletin, vol. 40 (1934), pp. 413–417, Theorem 6) it is natural to raise the general question: what transformations of a sequence will preserve convergence in variation? Phrasing the problem in terms of complex functions of a real variable, it is proposed in the present paper to answer this question as fully as possible. An incidental result is a theorem on change of variable in a Lebesgue integral, generalizing earlier theorems of de la Vallée Poussin and Fichtenholz. Some applications of the main theorem are: convergence in length (see Adams and Lewy, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26) is an additive and multiplicative property of sequences whenever *one* of the limit functions is absolutely continuous (extending Theorem 6 of Adams and Lewy); a necessary and sufficient condition for convergence in length is convergence in variation plus either convergence in measure or "almost convergence in the mean" of the derivatives (the latter type of convergence is intermediate between convergence in measure and convergence in the mean). The paper concludes with a necessary and sufficient condition for convergence in the mean and several generalizations of a theorem of Plessner (Journal für Mathematik, vol. 160 (1929), pp. 26–32, Theorem 1). (Received September 26, 1936.)

404. Dr. F. J. Murray (National Research Fellow) and Professor John von Neumann: *On rings of operators.* II.

This paper is a continuation of one by the same authors, *On rings of operators*, Annals of Mathematics, vol. 37 (1936), pp. 116–229. The authors investigate further the properties of factor rings in case II. It is shown that there exist elements $g_1, \cdots, g_m$ such that the trace $Tr_{\mathfrak{M}}(A)$ is expressible as $\sum_{i=1}^{m} (Ag_i, g_i)$. This implies the linearity and weak continuity of the trace. If $\mathfrak{M}'$ is also in case II_1 and in the standard normalization $C=1$ (cf. loc. cit., Theorem X), then m may be taken as 1 and $\mathfrak{M}$, $\mathfrak{M}'$ and a subset $\mathfrak{A}$ of $\mathfrak{H}$ are isomorphic. It is possible to complete $\mathfrak{M}$ (and $\mathfrak{M}'$) by adding operators A which are $\eta\mathfrak{M}$ (or $\eta\mathfrak{M}'$) so that the resulting set is isomorphic to $\mathfrak{H}$. Finally it is shown that if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are factors satisfying these conditions then an algebraic isomorphism implies an inner spatial isomorphism between them. (Received September 3, 1936.)

405. Dr. G. B. Price: *On the extreme points of convex sets.*

This paper investigates the properties of extreme points of convex sets in spaces which are linear and normed. If the metric has the property that the segment joining any two points of the spherical neighborhood $S(x, r)$ is interior to the neighborhood except at most for the two points themselves, the space is denoted by L^*, and in all other cases by L. The study of extreme points is far simpler in spaces L^* than in spaces L, and the results are more complete. The results obtained include the following: An approximation theorem first proved by Minkowski for euclidean 3-space is generalized in two ways for spaces L and L^*. If a set G in a Banach space can be approximated in a specified manner by compact sets, then G itself is compact. These two results lead to a new proof

that the convex hull of a compact set in a Banach space is compact. The closed convex hull of the set of extreme points of a closed, compact, convex set in a space L^* is the convex set itself. Furthermore, the relation between a set S and the extreme points of its closed convex hull and also the extreme points of the limit set of a sequence of closed, compact, convex sets are studied in detail. (Received September 9, 1936.)

406. Dr. W. T. Reid: *On the Jacobi condition for the double integral problem of the calculus of variations.*

This paper is concerned with the Jacobi condition for the problem of minimizing $I=\iint_A f(x, y, z, z_x, z_y)dxdy$ in a class of admissible surfaces $z=z(x, y)$ with fixed values on the boundary C of A. The usual form of the Jacobi condition states that along a non-singular minimizing extremal surface E there can exist no solution $u(x, y)$ of the accessory equation defined in a neighborhood of a simply closed regular curve C_0 and its interior A_0 in A, with $u\equiv 0$ on C_0 and $u\not\equiv 0$ in A_0. It is proved in the present paper that the above condition is still true when the phrase "simply closed regular curve C_0 and its interior A_0" is replaced by "connected open set $\mathcal{A}$ and its frontier $\mathcal{C}$," "A_0" by "$\mathcal{A}$" and "C_0" by "$\mathcal{C}$." When E is not an extremal, but of class C', a corresponding theorem is proved in terms of the Haar form of the accessory equations. The formulation of the Jacobi condition in this latter case, however, involves the additional assumption $|u_x|+|u_y|\not\equiv 0$ on $\mathcal{C}$. The extended form of the Jacobi condition here proved is of use in the study of the associated accessory boundary problem. The principal results of this paper were obtained in the summer of 1933, on the occasion of Professor Bliss' lectures at the University of Chicago on multiple integral problems of the calculus of variations. (Received October 3, 1936.)

407. Mr. L. B. Robinson: *A determinant used in functional equations.*

Consider the determinants of infinite order $\Delta^{(n-1)}\equiv|\delta_{ij}+a_{ij}|$, where $\delta_{ii}=1$, $\delta_{j+1,j}=S_1+S_2+S_3+\cdots+S_{n-1}$, $\delta_{j+2,j}=S_1S_2+S_1S_3+\cdots+S_{n-2}S_{n-1}$, $\cdots$, $\delta_{j+(n-1),j}=S_1S_2\cdots S_{n-1}$, $|S|<1$. The other δ's vanish. Extract from the above the upper left-hand determinant of $p+r$ and p rows respectively, that is, $\Delta_{p+r}^{(n-1)}$, $\Delta_p^{(n-1)}$. Write the inequality $|\Delta_{p+r}^{(n-1)}-\Delta_p^{(n-1)}|<\epsilon$, $r=1, 2, 3, \cdots$, where ϵ is as small as one pleases and n is greater than some arbitrary number. If this inequality be always satisfied then $\Delta^{(n-1)}$ converges. It can be proved by addition and subtraction of rows that $\Delta_{p+r}^{(n-1)}\equiv\Delta_{p+r}^{(n-2)}\equiv\cdots\equiv\Delta_{p+r}^{(0)}$ the last being the normal determinant of Poincaré. The same is true of Δ_p. But $|\Delta_{p+r}^{(0)}-\Delta_p^{(0)}|<\epsilon$. The author has proved the convergence of $\Delta^{(n-1)}$. (Received September 29, 1936.)

408. Dr. A. R. Schweitzer: *On the definition of "sense" in the foundations of geometry.*

This paper is in continuation of *A theory of geometrical relations*, American Journal of Mathematics, 1909, 1913. Dr. Schweitzer gives four definitions of "sense" in the foundations of geometry and discusses their mutual interrela-

tions. The first definition is an n-dimensional generalization ($n=1, 2, 3, \cdots$) of the definition given by the author in the Transactions of this Society, 1909, p. 313. The second is an application of the abstract definition given by the author in the American Journal of Mathematics, 1909, p. 372. The third is phrased in terms of the author's relations B_n and S_n, as indicated in the American Journal of Mathematics, 1909, p. 366. The fourth definition is topological and is based on the author's relation I_n. These four definitions correspond roughly to standpoints in the theory of definitions of "sense" of (1) Pieri, (2) Hilbert, (3) Thaer, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 28 (1919), and (4) Klein, Veblen, and J. R. Kline. (Received September 8, 1936.)

409. Dr. S. E. Warschawski: *On the behavior of a conformal mapping at a cusp.* II.

The present paper is a continuation of the preliminary report given in abstract 42-5-219. Let C be a closed Jordan curve in the w-plane with a cusp at $w=0$ which is characterized by a function $\phi(\rho)$ as described in that abstract. The following are the results obtained in this paper: 1. Let $z(w)$ map the interior R of C conformally on $|z+1|<1$ and let $z(0)=0$. If there exists an increasing function $\mu(\rho)$ such that $|d\phi(\rho)/d\rho| \leqq \mu(\rho)$ and $\int_0^\alpha \mu(\rho)^2\rho d\rho/\phi(\rho)$ converges, then, for $|w|=r$, $\lim_{r\to 0}[|z(w)|/\exp(-\pi\int_r^\alpha d\rho/\rho\phi(\rho))]=\lambda$ exists in $R+C$ and λ is *positive*. 2. Let $z(w)$, instead of representing R on a circle, map R on another region R' of the z-plane, bounded by a closed Jordan curve C' with a cusp at $z=0$, which is characterized by a function $\psi(\rho)$ defined in the same manner as $\phi(\rho)$. A *necessary and sufficient* condition, involving $\phi(\rho)$ and $\psi(\rho)$ only, is given in order that the "dilatation" of the map at $w=0$, that is, $\lim_{w\to 0}|z(w)|/|w|$, be a *definite finite* number. (Received October 3, 1936.)

410. Professor Hassler Whitney: *On the maps of an n-sphere into another n-sphere.*

An elementary and simple proof is given of the fact that two maps of one n-sphere into another n-sphere are homotopic if and only if they have the same degree. (Received September 26, 1936.)

411. Professor Kurt Friedrichs: *On certain inequalities and characteristic values problems for analytic functions and for functions of two variables.*

In the first part of the paper the author establishes an inequality for $\iint w^2dxdy$ under the conditions $\iint |w|^2dxdy<\infty$ and $\iint wdxdy=0$. This inequality leads to a complete solution of the reduction of the quadratic form $\iint w^2dxdy$ relative to the form $\iint |w|^2dxdy$. The rexults of the first part are applied in the second part to the theory of equilibrium and vibration of an elastic plate. (Received October 10, 1936.)

412. Dr. E. L. Post: *Finite combinatory processes. Formulation* 1.

The present formulation envisages a general problem consisting of a class of specific problems. A symbol space is postulated in which each specific prob-

lem and corresponding answer can be represented, and in which the work leading from symbolized problem to symbolized answer is to be carried out. A fixed set of directions both directs operations in the symbol space and determines the order in which those directions are to be applied. It gives a solution of the general problem if on applying it to each specific problem the process thus set up terminates in the answer to that problem. The present formulation describes a specific symbol space, enumerates the primitive processes the problem solver is assumed capable of performing in that symbol space, and gives a definite structure to the set of directions. Its purpose is to present a model for formulations of increasing psychological complexity and a norm for their logical reduction, the whole to yield a theory of the limitations of mathematics along the lines already initiated by Gödel and Church. (Received October 31, 1936.)

413. Professor A. A. Albert: *A note on the matrices defining total real fields.*

A field K of degree n over a field F of real numbers is called total real if its complex conjugate fields over F are all fields of real numbers. The author gives a matric construction of all such fields by proving that K is total real over F if and only if K is equivalent over F to $F(Z)$, $Z=DS$, where D is an n-rowed diagonal matrix with diagonal elements positive numbers of F and S in any symmetric matrix with elements in F for which the characteristic function of DS is irreducible in F. The irreducibility condition is a usual final condition in problems on the construction of fields. (Received November 2, 1936.)

414. Professor A. A. Albert: *Normalized integral bases of algebraic number fields.* I.

A generation $F=R(\theta)$ of an algebraic number field F of degree n over the field R of all rational numbers is called primitive if θ has zero trace and 1, $\theta, \omega_3, \cdots, \omega_n$ form a basis of the algebraic integers of F. The author proves that every F has a primitive generation and obtains necessary and sufficient congruential conditions on the minimum function of θ that θ provide a primitive generation. These conditions are applied to cubic, quartic, and quintic fields to obtain explicit coefficient restrictions in these cases. The normalization to a primitive generation is applied to simplify the author's explicit determination of the integers of all cubic fields (Annals of Mathematics, 1930) and subsequently further normalizations are also obtained. It is then shown that every cubic field is equivalent to one of a set of certain classes of fields such that no field in any one class is equivalent to a field in any other. The classification is an invariantive one in terms of the discriminant of the field and certain other invariants, and the results give an explicit determination of the discriminant of every cubic field in terms of the coefficients of the minimum function in a normalized generation. (Received November 2, 1936.)

415. Mr. C. B. Allendoerfer: *The imbedding of Riemann spaces in the large.*

The classical theory of the imbedding of a Riemann space in a euclidean space has recently been revised in an elegant manner by W. Mayer (Trans-

actions of this Society, vol. 38, pp. 267–309). In Mayer's theory the imbedding is defined by a number of forms, but precise conditions which the coefficients of these forms must satisfy are not given. Part I of the present paper obtains these conditions in concise form and recasts the entire theory so that it applies to any simply connected Riemann manifold. In part II there is defined an arithmetic invariant, τ_p, of a Riemann space for every positive integer p. It is shown that if a space with $\tau_p \geqq 3$ can be imbedded such that p is the rank of its second fundamental form, then the imbedding is rigid. Similarly if $\tau_p \geqq 4$, it is shown that the Codazzi equations of the classical theory are algebraic consequences of the Gauss equations. (Received November 6, 1936.)

416. Dr. J. L. Brenner: *The normal subgroups of the groups of matrices mod p^r. II.*

G is the group of non-singular n^2 matrices mod p^r. G is isomorphic with the group of automorphisms of the abelian group of order p^{nr} and type $(r, r, \cdots, r)$. All the normal subgroups of G are found when $p=2$. If $n \neq 2$, the results parallel those when $p>2$ (see abstract 42-7-275). Let $s((a_{ij})) = \min_{i \neq j} (a_{ij}, a_{ii}-a_{jj}, p^r)$. Normal subgroups N are classified according to N', the subgroup of N of determinant 1 (p^r), and N'', the subgroup of N' which satisfies $a_{ii} \equiv 1 (p^s)$, $s = \min s(A)$, $A \in N$. The classification involves four natural parameters, or five if $n=2$. (Received November 2, 1936.)

417. Mr. J. F. Daly: *On the points of an algebraic manifold not reachable by a given parametric representation.*

It is well known that a given parametric representation of an irreducible algebraic manifold will not in general cover the manifold completely. This paper shows, however, that if the base field is the complex field, than any point of the associated manifold not obtainable by an allowable specialization of the given parameters is always a limit point of points which are thus obtainable. The proof consists essentially in choosing a new transcendental basis for the field defining the parametric representation, and then employing a theorem of Ritt. (Received October 29, 1936.)

418. Professor W. L. Duren: *Contractible problems of Bolza.*

Suppose that the functions which define admissible arcs for a non-parametric problem of Bolza in the calculus of variations can be divided into two groups, $y_h(x)$, $(h=1, \cdots, r)$ and $z_s(x)$, $(s=1, \cdots, l)$, $(r+l=n)$, in such a way that the side conditions determine a one-to-one correspondence between the admissible sets $y_h(x)$, $z_s(x)$ and the sets $y_h(x)$. Then the problem is said to be contractible. For the case that the end conditions do not involve the end values of the functions $z_s(x)$ this paper develops conditions which are sufficient to insure that an extremal arc E furnishes a minimum with respect to arcs having points (x, y) neighboring those of E. The fields used are based upon an invariant integral in xy-space and are constructed from families of extremals all of which satisfy the conditions $F_{z's}=0$ identically. A Weierstrass condition stronger than the conventional one is required. (Received November 6, 1936.)

419. Professor Arnold Emch: *On cremona involutions connected with the Weddle surface and some covariant surfaces.*

This paper deals with involutorial cubic and Geiser transformations T_3 and T_7 connected with the Weddle surface. It is shown that the product of T_3 and T_7 is a T_9 of order nine with the property $T_9 = T_7T_3 = T_3T_7$. The properties of T_9 lend to a new covariant sextic surface F_6 which is pointwise invariant in T_9. Finally a parametric representation of F_6 and similar surfaces is given. (Received October 28, 1936.)

420. Mr. D. W. Hall and Dr. G. E. Schweigert: *On non-n-alternating transformations.*

A study is made of non-alternating transformations $T(A) = B$ on compact connected sets where the transformation satisfies either of the conditions: (1) if for any point x of B there exists a cutting Σ of $A - T^{-1}(x)$ consisting of at most n points, then there is no point y of B such that the inverse of y intersects both sets of the separation $A - [T^{-1}(x) + \Sigma]$; (2) for any component K of $B - x$, the inverse of K is not separated by the removal of any n of its points. Transformations of the second type are also of the first type on locally connected sets. Let A be a set every two points of which are separated by at most k points. If B is non-degenerate and $k \leqq n$, then T is not of the second type; furthermore T is of the first type if and only if it is homeomorphic; while if $k = n+1$, every T of the first type is A-set reversing (see G. T. Whyburn, American Journal of Mathematics, vol. 58, p. 311) and each T of the second type sends every k cutting of A into a single point. (Received October 30, 1936.)

421. Professor C. G. Latimer: *The classes of integral sets in a quaternion algebra.*

A generalized quaternion algebra $\mathfrak{A}$ contains an infinitude of integral sets. All the sets equivalent to a given set are said to form a class. It is shown that the norm of the general element of trace zero in an integral set $\mathfrak{G}$ is a ternary quadratic form f, which belongs to a class uniquely determined by the class of sets containing $\mathfrak{G}$. Moreover, the genus G which contains f or $2f$, according as the fundamental number d of $\mathfrak{A}$ is even or odd, is uniquely determined by $\mathfrak{A}$. It is shown that there is a one-to-one correspondence between the classes of integral sets in $\mathfrak{A}$ and all the classes in G or all the improperly primitive classes in G, according as d is even or odd. From this and a known result on ternary forms, it follows that if $d < 0$, then any two integral sets in $\mathfrak{A}$ are equivalent. (Received November 5, 1936.)

422. Professor Tibor Radó: *On the harmonic majorants of subharmonic functions.*

Let $u(x, y)$ be subharmonic in a bounded domain (connected open set) G. Denote by D a Dirichlet domain comprised in G together with its boundary. For such domains D one has (F. Riesz, Acta Mathematica, vol. 48, p. 334) a *best harmonic majorant* h for u in D, defined solely in terms of the values of u on the boundary of D. Besides, F. Riesz has also considered a *least harmonic majorant* h^* of u in D, defined in terms of the values of u in D itself (Acta Math-

ematica, vol. 54, pp. 353–358). *The purpose of this paper is to prove that $\bar{h}$ is always equal to h^*.* As far as the author is aware, this was only known for special types of subdomains. The result can be stated in the following equivalent form which seems to express more adequately its true meaning. Denote by h a function which is harmonic in D and define in G a function v as follows: $v=h$ in D and $v=u$ in $G-D$. If this function v is again subharmonic in G, then let us say that h is admissible for u in D. One has then the following theorem. *If D is a Dirichlet domain comprised in G together with its boundary, then there exists exactly one harmonic function which is admissible for u in D.* (Received October 14, 1936.)

423. Dr. J. B. Rosser: *Gödel theorems for non-constructive logics.*

By "Carnap's Rule" shall be meant the rule which can be roughly stated: "If $f(0)$, $f(1)$, $f(2)$, $\cdots$, then (x) $f(x)$." The logics got by starting with the system of Principia plus Peano's axioms and allowing 1, 2, $\cdots$, ω, $\omega+1$, and so on up to, but not including, ω^2 uses of Carnap's Rule (interspersed with uses of the ordinary rules) are studied. It is proved that if one of these logics is simply consistent, then this logic contains undecidable propositions, does not contain the formula stating the simple consistency of the logic, and is not closed under Carnap's Rule. (Received October 20, 1936.)

424. Dr. W. E. Sewell: *A note on the development of functions in a sequence of normal and orthogonal polynomials in the complex domain.*

An application of recent results of the author (Proceedings of the National Academy of Sciences, vol. 41 (1935)) and a method due to Dunham Jackson (this Bulletin, vol. 36 (1930)) lead to the following result: Let C be a rectifiable Jordan curve in the z-plane and let $P_n(z)$ be an arbitrary polynomial of degree n in z. Let $\int_C |P_n(z)|^p |dz| = \epsilon_n$, $p>0$. Then $|P_n(z)| \leqq M n^{2/p} \epsilon_n^{1/p}$, z on C, where M is a constant independent of n and z. Let $\{P_n(z)\}$ be the set of polynomials normal and orthogonal on C and let $f(z)$ belong to L^2 on C, and let $f_1(z) = a_0P_0(z) + a_1P_1(z) + \cdots + a_kP_k(z) + \cdots$, $a_k = \int_C f(z) \overline{P_k(z)} |dz|$. Now by putting $\epsilon_n = 1$ and $p=2$ in the above inequality the following result is obtained: If $\sum_{k=0}^{\infty} |a_k| k$ converges, the function $f_1(z)$ is analytic in C, continuous in $\bar{C}$, the closed limited set bounded by C, and $f_1(z) = f(z)$ almost everywhere on C. (Received October 27, 1936.)

425. Dr. C. B. Tompkins (National Research Fellow): *Some integer-valued integral invariants of closed curves and bands.*

A band is a homeomorph in three-space of the portion of a plane between two concentric circles. A contour is a simple closed curve which can not be shrunk on the band to a point. The Gauss integral gives the number of interlinkings of two nonintersecting contours on the band. This integral is split into the sum of two integrals, one depending only on a contour, the other depending on surface-theoretic quantities along this contour (it is a

known integral of G. Y. Rainich). These two integrals are integer-valued. Another integral is introduced, and when added to the Rainich integral it gives an integer which is invariant under locally isotopic deformations. This last integer is the number of local interlinkings of two adjacent contours. When this number is subtracted from the Gauss integral there is left an integer, depending only on the contour of integration, which gives the degree of knottedness of the band or of its contours. (Received November 5, 1936.)

426. Professor T. L. Wade: *Algebraic concomitants by tensor algebra: Reduction.*

In this paper it is shown how certain identities satisfied by the numerical tensors may be used in the reduction of and in the establishment of syzygies on algebraic concomitants constructed by composition of tensors. The fundamental identities are given for the binary domain subjected to the unrestricted linear transformation, and for the ternary domain subjected to the unrestricted linear transformation, and also for the restricted linear transformations (affine and euclidean). While such identities are closely analogous to those of the second fundamental theorem of the Clebsch-Aronhold symbolic method, there are distinctions; for one thing the identities employed here are on real vectors and matrices satisfying the tensor law of transformation as in contrast to the identities of the symbolic method which are on "ideal" vectors, whose elements in general have no particular meaning except when they occur in products involving a specified number of them. (Received November 2, 1936.)

427. Professor T. L. Wade: *An irreducible complete system by tensorial methods of euclidean concomitants for the line and conic.*

By using the method of constructing concomitants by composition of tensors and the tensor identities discussed by the author in a previous paper, an irreducible complete system of euclidean concomitants for the line and the conic is found. This system consists of forty-two concomitants: eight invariants, ten covariants, fifteen contravariants, and nine mixed concomitants. Of these concomitants twelve are algebraically independent; so the forty-two concomitants are connected by thirty syzygies. This set of thirty syzygies is established with the aid of the fundamental identities used in reduction. (Received November 2, 1936.)

428. Professor J. L. Walsh and Professor G. M. Merriman: *Note on the simultaneous orthogonality of harmonic polynomials on several curves.*

Szegö has recently determined (Transactions of this Society, vol. 37 (1935), pp. 196–206) sets of polynomials, $p_n(z)$, in the complex variable z which are simultaneously orthogonal with respect to suitable norm functions on all level curves of a given family. In the present note the authors show that the harmonic polynomials found by separating the $p_n(z)$ into real and pure imaginary parts are also orthogonal on the corresponding level curves, except in Szegö's case II with $\alpha>2$. (Received May 27, 1936.)

429. Mr. A. L. Whiteman: *On a set of postulates for Boolean algebras by terms of triadic rejection.*

The operation of triadic rejection in Boolean algebra is the operation ( ) given by $(abc)=a'b'+b'c'+c'a'$. It is the object of this paper to present a set of postulates for Boolean algebras in which triadic rejection is taken as the only primitive idea, besides that of class. As a result, all the special Boolean elements are introduced with an elegance not possible in any other set known to the author. Thus the negative of an element is defined in terms of the primitive ideas, and then any two contradictory elements are chosen to represent the zero element and the universe element of the resulting Boolean algebra. (Received October 29, 1936.)

430. Professor Hassler Whitney: *Matrices of integers and combinatorial topology.*

Two types of conditions for the solubility of equations in integers $\sum a_{ij}x_j = y_i$ (or with x_j, y_i elements of an abelian group) are given. The proof is based on a general theorem on the extension of a homomorphism. Cocycles and coboundaries, dual to cycles and boundaries, are characterized in terms of the latter, and vice versa. A theorem relating homology and cohomology groups is given. As an application, a simple basis for the theory of closed and irreducibly closed complexes is given. (Received October 27, 1936.)

431. Dr. J. M. Feld: *Counterpoints and associated cubics.*

Properties of the projective counterpoint (isogonal) transformation and their bearing on the geometry of plane cubic curves are discussed. It is shown that on each line of the plane lies a pair of counterpoints and that the pairs of counterpoints on the lines of a sheaf of lines lie on a cubic curve. Starting with three pairs of counterpoints an infinite number of other pairs can in general be derived from them by straight-edge alone, and these derived pairs as well as the original ones all lie on a cubic curve. (Received October 15, 1936.)

432. Dr. D. T. Sigley: *Orders for which there exist exactly six groups.*

If one prime factor of a number n is congruent to unity with respect to another prime factor of the number n, it is said that n involves a *unity congruence* as defined by G. A. Miller (American Journal of Mathematics, vol. 55 (1933)). The number of abstract groups of a given order is a function of the number of distinct unity congruences in the order. With the aid of the results of the paper cited above, the orders for which there exist precisely six groups are determined; the previous cases have been determined by G. A. Miller (Proceedings of the National Academy of Sciences, vol. 19 (1933), two papers). As a final result an approximation to the asymptotic relation between the number of groups of a given order and the number of unity congruences in that order is given. (Received October 16, 1936.)

433. Mr. P. O. Bell: *A characteristic property of the directrix of Wilczynski.*

The directrix of the first kind which Wilczynski introduced intersects the

tangents to the asymptotic curves of a surface in two covariant points. In this paper a characteristic property of these points is given which associates them with a number of covariant points whose general coordinates are obtained by a sequence of partial differential operations. (Received October 23, 1936.)

434. Professor G. D. Gore: *Transformations of a surface bearing a family of asymptotic curves.*

Two transformations are established for any non-developable surface that bears a family of asymptotic curves and is immersed in a space of n dimensions ($n<3$). The first is an analogue of the transformation of Levy for conjugate nets. The second is an analogue of the fundamental transformation F of Eisenhart. (Received October 24, 1936.)

435. Professor W. G. Warnock: *Triple systems as ruled quadrics.*

The triple of a triple system is considered as a ruled quadric. The number and kinds of intersections of ruled quadrics so formed are studied. (Received October 26, 1936.)

436. Dr. G. M. Ewing: *Sufficient conditions for a non-regular problem in the calculus of variations.*

Usual sufficient conditions for the problem of minimizing $J=\int f(x, y, y')dx$ include the condition (III') that $f_{y'y'}(x, y, y')$ shall not vanish on the minimizing curve. The present paper assumes an arc $E: y=e(x)$ satisfying the necessary conditions of Euler, Weierstrass, and Legendre (I, II, and III). The one parameter family of integrals $L=\int\phi(x, y, y')ds$, $\phi=f(x, y, y')+k^2 \cdot [y'-e'(x)]^2$, $k\neq 0$, is then introduced. E is an extremal for L and a further necessary condition for E to minimize J is that it satisfy the Jacobi condition in the strict form (IV') for L. Conditions sufficient for E to furnish an improper strong relative minimum for J are obtained by suitably strengthening these necessary conditions. (Received October 26, 1936.)

437. Dean E. B. Stouffer and Mr. E. W. Emery: *Note on the characterization of Sannia's tetrahedron O.*

In the study of projective differential properties of a curve in ordinary space use is made of a linear homogeneous differential equation of the fourth order. The Halphen canonical form of this differential equation is analytically the simplest canonical form known. Its associated tetrahedron of reference has been located geometrically by Sannia by the use of certain dual properties. In this paper the introduction of dual coordinates is avoided by means of a uniquely determined section of the developable surface generated by the tangents to the curve. (Received October 26, 1936.)

438. Dr. V. P. Jensen and Professor D. L. Holl: *An application of non-analytic functions in plane stress problems.*

In two dimensional plane stress problems, Airy's stress function may be written as a function of the conjugate variables z and $\bar{z}$ so that $F(x, y)$

$=F[(z+\bar{z})/2, (z-\bar{z})/2i]$. It is shown that for the non-analytic function $H=2\partial F/\partial\bar{z}$, the Kasner circle defined by the directional derivative of H is identical with Mohr's circle of stress. Additional properties follow by which one is enabled to discuss the nature of the stresses on curvilinear boundaries as well as the nature of the stresses for the boundary of a new problem obtained by conformal mapping. (Received October 26, 1936.)

439. Dr. M. S. Webster: *Orthogonal polynomials with orthogonal derivatives.*

W. Hahn has proved that orthogonal polynomials which possess orthogonal derivatives are essentially Jacobi, Laguerre, or Hermite polynomials. More recently, Krall gave a new proof of the theorem when the interval of orthogonality is finite. This paper extends this method to the case when the interval is infinite, thus obtaining the Laguerre and Hermite polynomials. (Received October 26, 1936.)

440. Professor W. S. Kimball: *Derivatives of line integrals.*

Formulas for the derivatives of line integrals are developed by a Δ process similar to that employed in deriving the new critical equations for extrema in the calculus of variations. An important advance is made in that attention is not restricted to cases where the integrals are maxima and minima, but expressions are derived for their partial rates of change with respect to x and y at any point on every path of integration admitted to consideration, it being noted how these rates of change, both first and second derivatives, are characterized along the "extremal" arcs. The cases treated are line integrals with fixed integration limits, those with variable upper limit and fixed lower limit, and those with fixed upper and variable lower limit. Derivative formulas are derived not only with respect to variable limits of integration, but with respect to x and y at any point along any path between these limits. Second and higher derivatives of line integrals are also discussed, but attention is restricted to single integration. (Received October 27, 1936.)

441. Professor Arnold Emch: *On symmetric ruled surfaces.*

This paper deals with symmetric ruled and developable surfaces and their properties. In particular it is shown how such surfaces are obtained by their proper analytic representation. Some typical examples are given to illustrate the theory. (Received October 27, 1936.)

442. Professor A. S. Householder: *The dependence of a focal point upon curvature in the calculus of variations.*

Methods of the tensor analysis for a Finsler space of n dimensions as developed by Cartan in a recent monograph are employed to develop a new normal form of the Jacobi equations with $n-1$ instead of n dependent variables. These equations are utilized to generalize results of Bliss for 2-space and of White for 3-space showing that the focal point of a curve transversal to an extremal varies monotonically on this extremal from its intersection with the curve, as the curvature of the curve at this intersection decreases from infinity. Moreover, the focal point approaches the first point conjugate to this inter-

section as the curvature approaches a definite limiting value which is finite or negatively infinite according as the transversal curve is or is not tangent to a certain linear subspace whose dimensionality is less than $n-1$ by the order of the conjugate point. (Received October 27, 1936.)

443. Dr. A. R. Schweitzer: *Projective analogues of descriptive systems of axioms in the foundations of geometry.*

A system of axioms which fall into two categories, (1) primarily without, and (2) with, reference to order, is constructed for projective geometry. In the first category the systems ${}^nK_n^*$ $(n=1, 2, \cdots)$ of axioms which are the analogs of the systems ${}^nK_n$ of a previous paper (American Journal of Mathematics, vol. 31 (1909), pp. 365–411) are constructed. The former systems are readily obtained from the latter with only two major changes. The systems ${}^nK_n^*$ are sufficient for projective geometry provided axioms are added which imply, under the axioms, the "fundamental theorem" of projective geometry. Also in the first category are constructed projective analogs of descriptive systems due to Veblen (Transactions of this Society, vol. 5 (1904), pp. 343–384) and the author (Transactions of this Society, vol. 10 (1909), pp. 309–314). In the second category fall the systems ${}^{n+1}K_n^{(0)}$ which are obtained from the systems ${}^nK_n$ by "tactical adjunction" (American Journal of Mathematics, vol. 34 (1912), p. 173). For n equal to 1 the system is properly projective. This system is extended to the n-dimensional system ${}^{n+1}K_2^{(0)}$. The latter system is then enlarged so as to provide complete sets of axioms for projective geometry in terms of a single undefined relation between ordered triads of points. (Received October 28, 1936.)

444. Professor N. H. McCoy: *Some theorems concerning non-associative rings.*

This paper presents a study of non-associative rings with unit element, special attention being given to the ideals in such rings. If M is a right ideal in the ring R then clearly R and M may be considered as additive groups with the left operator domain R. One of the principal results is an extension of a theorem of Noether and Schmeidler (Mathematische Zeitschrift, vol. 8 (1920), p. 11) concerning the relation between an additive decomposition of R/M and the expression of M as the L.C.M. of a "total teilerfremd" system of ideals. Special attention is also given to the study of two-sided ideals, in which case some of the decompositions are shown to be unique under assumption of the maximum and minimum conditions for two-sided ideals. (Received October 28, 1936.)

445. Professor Gordon Fuller: *Transformation of a system of partial differential equations into tensor form.*

This study represents an extension of an investigation reported before the Society on April 8, 1932 (abstract 38-5-128). A system of equations which has the same general structure as before is considered. The equations have four independent variables, however, instead of three, and are homogeneous in the derivatives. As before, the independent variables are not vector or tensor components; but new variables are introduced which may be considered as the

components of a vector, and new equations are established in terms of this vector which are consequences of the original system. The new system is of the second order, and non-linear. (Received October 29, 1936.)

446. Professor C. C. Camp: *Expansions involving differential equations in which the coefficient of the parameter changes sign.*

In the equation $X'+\lambda a(x)X=0$ the coefficient $a(x)$ is real and integrable such that $\int_{-1}^{1}a(x)dx=A(x)]_{-1}^{1}=2A_1\neq 0$, but otherwise unrestricted Hence it may vanish over a whole subinterval and at any number of other points. Let $f(x)$ be made up of a finite number of pieces, each real, continuous, and possessing a continuous derivative. For the boundary condition $X(-1)=X(1)$ the resulting expansion will converge, but not always to the value of f. Under certain conditions it will converge to f over a whole subinterval. Elsewhere it may converge to the usual mean value plus integral multiples of values of f at other points. A similar situation obtains at boundary points and points x_i where $A(x_i)-A(-1)=k_iA_1$ and k_i is an integer. The theory is extended to the more general case $X'+[\lambda a(x)+b(x)]X=0$, $X(-1)=cX(1)$, as well as to a system in p parameters. (Received October 29, 1936.)

447. Dr. D. G. Bourgin and Dr. R. Duffin: *A method for certain two-point boundary value problems.*

This method is distinct in principle from that of the Green's functions. In the cases to which it applies, a kernel $G(x, n)$ exists such that $MG(x, n)=N(n)\cdot G(x, n)$ where M is the operator adjoint to the original L, and $N(n)$ is a polynomial (though extensions to a linear operator for $N(n)$ are possible). For a suitable domain $\{\phi(x)\}$ defined on an x range properly containing the interval ab, a resolvent kernel $H(n, x)$ and a fixed path l exists such that $\phi(x^+)+\phi(x^-)=2\int_l\int_a^b H(n, x)G(n, t)\phi(t)dtdn=\int_l H(n, x)g(n)dn$. The function $g(n)$ is fixed in terms of the bilinear concomitant $N(n)$, and $\int_a^b G(x, n)f(x)dx$. In conjunction with the boundary relations $U_i(a, b)=\alpha_i$, $i=1, \cdots, n$, the stipulation that $g(n)$ associate an image function vanishing outside of $a\leqq x\leqq b$ determines a solution of the differential system for permissible $f(x)$. For constant coefficient equations $G=e^{-xn}$, $H=e^{xn}$, and the resolvent integral is that of Mellin. The $g(n)$ condition is that it be entire. (Received November 2, 1936.)

448. Professor W. H. Roever: *The deviating force on a rapidly moving aeroplane due to the earth's rotation.*

In order to account for the motion of a body with respect to a frame of reference fixed to the solid part of the earth, two forces are brought into play in addition to those which account for the motion with respect to an inertial frame of reference. These are the centrifugal force and compound centrifugal force. The resultant of the first of these with the earth's gravitational force constitutes weight. In the field of weight the second of these forces causes a moving body to deviate from its direction of motion. It is the purpose of this paper to determine this deviating force for a heavy rapidly moving aeroplane. (Received November 9, 1936.)

449. Dr. Hans Lewy: *A priori limitations for solutions of Monge-Ampère equations.* II.

By means of a regularizing contact transformation that preserves the Monge-Ampère type of the equation a certain convergence theorem for solutions of analytic elliptic Monge-Ampère equations is reduced to the result of first part of this paper (Translations of this Society, vol. 35 (1935), pp. 417–434). Additional conditions granting the analyticity of the limit function are established. (Received October 20, 1936.)

450. Dr. M. R. Hestenes: *A direct sufficiency proof for the problem of Bolza in the calculus of variations.*

In the present paper a direct sufficiency proof is given for the problem of Bolza which is independent of assumptions of normality. Heretofore sufficiency proofs for the problem of Bolza independent of assumptions of normality have been indirect in the sense that they depended on a transformation of the problem into an equivalent one of somewhat different form. No transformation is needed in the present paper. The relation between the present method and the earlier ones is also discussed. (Received October 26, 1936.)

451. Professor A. A. Shaw: *On Measures and Weights by Epiphanius.* II.

For the first part of this paper see abstract 42-5-247, and National Mathematics Magazine, vol. 11 (1926), pp. 3–7. This second part deals with a new analysis of the character of Epiphanius based upon a study of his extant letters to Bishop John of Jerusalem and to Saint Jerome; and of the works of his contemporaries (Saint Jerome, Socrates Scholasticus, Salaminus Hermias Sozomen, Theodoret, and others) as well as those of modern critics such as R. A. Lipsius, Phillip Schaff, and Bardenhewer. (Received October 26, 1936.)

452. Professor P. N. Daus: *The ternary cubic representing norm and its hessian.*

If $N(x, y, z)$ represents the norm of a cubic algebraic integer, and H is the hessian of $N=0$, it is shown that $H=2DN$, where D is the discriminant of the cubic irrationalities used as a basis. (Received October 28, 1936.)

453. Professor W. M. Whyburn: *Note on functional equations of a cyclic type.*

The functions determined by equations of the form $L(y_i)=\sum_{j=1}^{n}A_j(x)y_{\alpha i+\beta j+\gamma}$, where L is a linear operator and α, β, γ, n, are integers, constitute a highly useful group of functions. A former paper (this Bulletin, vol. 36, pp. 863–868) treated the case of these equations where $\alpha=1$. The present paper studies cases that arise when $\alpha<1$. The special case $\alpha=n-1$ is treated in detail and is shown to be equivalent to a pair of equations of the type $L(u)=B(x)\bar{u}$, $L(\bar{u})=\overline{B}(x)u$, where u and B are complex functions of the real variable x and $\bar{u}$ is the conjugate of u. Known results for second order, self-adjoint differential equations are used in a study of cases that arise when $L(y_i)=dy_i/dx$. (Received October 28, 1936.)

454. Professor Morgan Ward: *The ranks of apparition of primes in a linear divisibility sequence.*

The sequence of rational integers $(u): u_0, u_1, u_2, \cdots$, is said to form a linear divisibility sequence if it satisfies a linear recursion relation with integral coefficients and if u_n divides u_m whenever n divides m. We here attack the problem of determining a priori all terms of (u) divisible by a pre-assigned modulus. We give a complete solution of this problem by the methods of ideal theory for the important case of prime moduli dividing neither the discrminant of the characteristic polynomial of degree k associated with the recurrence nor the persymmetric determinant $\Delta(u)$ of order k built from the first $2k-1$ terms of (u). The problem is essentially deeper than the previously treated problems of determining the period and numeric of any linear sequence (Transactions of this Society, vol. 35 (1933): pp. 600–628) in that the nature of the group of the characteristic polynomial must be taken into account. (Received October 30, 1936.)

455. Professor Glenn James: *A higher upper limit to the parameters in Fermat's equation,* $x_1^n + x_2^n = x_3^n$.

This paper treats the so-called first case of Fermat's Last Theorem, in which the parameters x_i are not divisible by n. It has been previously proved that this case of the theorem holds for all $x_1 \leqq n(cn+1)^n$ where c is even and $x_1 < x_2 < x_3$. The present paper proves that c contains n as a factor. Since the theorem has been proved for $n = 14000$, it is now seen to be true for all values of n provided $x_1 \leqq 14000\ (312{,}000{,}000)^{14000}$. (Received October 30, 1936.)

456. Professor A. D. Michal and Mr. D. H. Hyers: *Differential invariants in an abstract differential geometry.*

In this paper the authors continue their study of abstract normal coordinates in a general differential geometry (see abstract 42-9-348). An existence theorem is proved for a certain functional equation in an abstract euclidean space. For the definition of such a space see A. D. Michal, I. E. Highberg, and A. E. Taylor, *Abstract euclidean spaces with independently postulated analytical and geometrical metrics* (accepted for publication by Tonelli's Annali di Pisa). The theorem is used to show that normal coordinate methods may be applied to the absolute calculus of forms in *covariant* as well as in contravariant vector fields. There results a replacement theorem for abstract differential invariants. (Received October 30, 1936.)

457. Mr. E. W. Paxson: *Existence theorems for ordinary differential equations in topological spaces.* Preliminary report.

In this note the author continues the discussion of differential equations in a non-metric linear topological space, L (see abstracts 42-9-355, 42-5-137). The problem of Cauchy is investigated for the equation $y'(\mu) = f(y, \mu)$ where $f(y, \mu)$ is a continuous function on a region in the topological space L, and on a real interval, to the space L. (Received October 31, 1936.)

458. Dr. A. E. Taylor: *Analytic functionals defined in Hilbert space.* Preliminary report.

In this paper the abstract theory of analytic functions previously announced (abstracts 41-11-424, 42-3-147, and 42-9-369) is applied to the study of analytic complex functionals $f(z)=f(z_1, z_2, \cdots)$ defined on open subsets of the complex Hilbert space H_0. The explicit form of the power series expansion of $f(z)$ about a regular point is obtained. The homogeneous polynomials which form the series are forms of the nth degree in an infinite number of variables, bounded in the Hilbert sense, with coefficients given in terms of the partial derivatives of $f(z_1, z_2, \cdots)$, evaluated at the point in question. By an example of the above sort it is demonstrated that the author's theory is more general than that of R. S. Martin (see, for instance, Comptes Rendus, vol. 197 (1933), p. 735), in that, although the two definitions of analyticity are locally equivalent, the radius of analyticity about a given point, as defined by Martin, is in certain cases smaller than as defined by the author, whereas all fundamental properties of the function obtain in the larger region. (Received October 31, 1936.)

459. Professor Max Zorn: *A remark on real division algebras.*

The author investigates the consequences of a generalized associative law in division algebras over the field of all real numbers. (Received October 31, 1936.)

460. Professor E. T. Bell: *Certain ternary cubic arithmetical forms.*

By an obvious oversight, G. B. Mathews (*Proceedings of the London Mathematical Society*, vol. 21 (1891), pp. 280–287) gave an incomplete statement of a special case of Dirichlet's theorems on representations by norms of algebraic integers. Rectifying the oversight, we find forms for which the number of representations of any integer is finite. For example, 9 p, where p is a prime $6n+1$, has precisely 18 representations in $x^3+y^3+z^3-3xyz$. (Received November 2, 1936.)

461. Mr. Victor Elconin: *Composite operators.*

Let S be a set of sets. For $\sigma \subset S$ let $\mu\sigma$ be the union of the elements of σ. For any *operator* T in S, $T\sigma$ is a function on a subset of S to S. The *range* ρT and the domain δT of T are those of $T\sigma$. T is *composite* if (i) $\sigma \subset \mu\delta T$ implies $\sigma\epsilon\delta T$, and (2) $\sigma\epsilon\delta T$ implies $T\sigma$ is the union of the set of all Tf for $f\epsilon\sigma$. Only composite operators are considered here. T is *simple* if $f\epsilon\mu\delta T$ implies $Tf\epsilon\mu\rho T$. Definitions of T^{-1}, if T is simple, and of TU are given. T is of *index* n if for any $s_1, s_2, \cdots, s_r$ in μS there is a unique $f\epsilon\delta T$ such that $f=s_1$, $Tf=s_2, \cdots, T^{n-1}f=s_n$. Linear operators, defined when $\mu\delta$ is a linear space, are studied in detail, and the results obtained are illustrated by special cases, including that of the differentiator $T=\sum_0^n a_k D^k$ in S where $\mu\delta$ is a function space. The entire theory of T^{-1} is obtained without recourse to Fuchsian existence or uniqueness theorems. (Received November 2, 1936.)

462. Dr. Hans Lewy: *Generalized integrals and differential equations.*

The notion of an integral $\int a(x_1 J(x))db(x_1 f(x))$, in which a and b are continuously differentiable functions of *two* variables and $f(x)$ is a function of Baire's first class, is introduced and certain continuity properties of it are proved. Similarly "solutions" of differential equations are defined and determined in the case that the equation involves the derivative of a function $f(x)$ of which no more is known than that it belongs to Baire's first class. The results obtained imply new types of à priori limitations for solutions of differential equations, and new existence theorems in the large. (Received October 20, 1936.)

463. Dr. Hans Lewy: *On differential geometry in the large.*

The results of the preceding paper are applied to Minkowski's and Weyl's problems: to determine closed convex surfaces whose curvature is positive and given as function of the direction of the normal (Minkowski) or whose metric is given (embedment problem in the large). In both cases existence of analytic surfaces solving the proposed problem is shown for the case of analytic data satisfying the necessary equalities. (Received October 20, 1936.)

464. Mr. Alvin Sugar: *Waring's problem for almost nth powers.* Preliminary report.

We shall call a member of the class $K_{nt}=[0, 1, x^n+t]$, where n is a positive integer and x and t are non-negative integers, an almost nth power of order t. The purpose of this paper is to evaluate $g(K_{nt})$, which is defined as the least number of almost nth powers of order t required to represent every positive integer. (Received October 20, 1936.)

465. Professor Einar Hille: *On bilinear formulas and Laplace transforms.*

This note contains a study of the application of bilinear series for the kernel $e^{-zu}\sim\sum_0^\infty \phi_n(u)\ \psi_n(z)$, $0<u<\infty$, $0<R(z)$, to the transformation of Laplace. Here $\{\phi_n(u)\}$ is a complete orthonormal system in $L_2(0, \infty)$. If $F(u)\sim\sum_0^\infty a_n \cdot \phi_n(u)$ and $f(z)=\int_0^\infty e^{-zu}F(u)\,du$, then $f(z)\sim\sum_0^\infty a_n \psi_n(z)$. If $F(u)\epsilon L_2(0, \infty)$, the first series is the Fourier orthogonal series of $F(u)$ and the second one converges absolutely to a function $f(z)\epsilon H_2(0)$. The relationship between the two series persists in more general cases, however, if $\{\phi_n(u)\}$ is suitably chosen, and can be used to obtain inversion formulas for the Laplace and Laplace-Stieltjes integrals. Special cases of such formulas have been found by F. Tricomi and D. V. Widder, using Laguerre polynomials, and by F. Hausdorff, using Legendre polynomials. (Received October 28, 1936.)

466. Profesor R. D. James: *On a problem in additive number theory.* Preliminary report.

A short time ago the author was asked by Professor D. N. Lehmer if there were any results on the representation of an integer as a sum of integers all of whose prime factors were of a certain linear form. The purpose of this paper is to give results of this type in two directions. First of all, it follows immedi-

ately from Schmielmann's general theorems (Iswestija Donskowo Polytechnitscheskowo Instituta (Nowotscherkask), vol. 14 (1930)) that every integer is a sum of a finite number of integers, all of those prime factors are of the form $ak+b$, where a and b are any relatively prime integers. Secondly, by considering certain quadratic forms, results can be obtained of which the following is an example. Every integer is a sum of at most 7 integers which are either 1 or have all their prime factors of the form $8k+1$. The number 7 cannot be replaced by any smaller number. (Received October 26, 1936).

467. Professor P. H. Daus: *A geometric interpretation of the convergents for a ternary continued fraction.* Preliminary report.

If θ is the root of an irreducible cubic equation of negative discriminant, we consider the point $P(x_n, y_n, z_n)$, where x_n, y_n, z_n is the nth convergent set of the ternary continued fraction determined by θ, and the family of syzygetic cubic curves $N(x, y, z)=t_nxyz$, where t_n is determined by the nth convergent set. Each curve is a two-circuited cubic, containing one and only one convergent point. The ovals form a non-intersecting nested set converging to the cubic irrationalities in question. This property is being made the basis of a study of the choice of the partial quotients in the ternary continued fraction expansion. (Received October 28, 1936.)

468. Professor Arnold Emch: *On certain configurations of points in space and linear systems of surfaces with these as base points.*

This paper deals with the configuration of points G_{27} of points W defined by $W(w^\alpha, w^\beta, w^\gamma, 1)$, $\alpha, \beta, \gamma \equiv 0, 1, 2$, (mod. 3), A_1 (1000), A_2 (0100), A_2 (0010), A_2 (0001); and the G_{36} of points V, which outside of the A's are the vertices of the 4 flex triangles of the syzygetic pencil of cubics of each face. Connected with these as base-points are linear systems of surfaces of which some typical examples are investigated. (Received October 29, 1936.)

469. Mr. D. H. Hyers: *Lebesgue integrals for functions out of a quasi-convex linear topological space.* Preliminary report.

The definition of Lebesgue integrals adopted by Garrett Birkhoff for functions with values in a Banach space (Transactions of this Society, vol. 38 (1935), p. 357) is extended to the case where the Banach space is replaced by a linear topological space. It is assumed that the space is complete and that every neighborhood satisfies a certain convexity condition. (Received October 31, 1936.)

470. Professor A. D. Michal: *Differential geometry of an abstract topological group with Banach coordinates.* Preliminary report.

This paper deals with a new chapter in the author's abstract differential geometries with coordinates in a Banach space. The geometry of Lie continuous groups studied by Cartan, Eisenhart, Schouten, and many others during

the last decade is brought under this more general theory. (Received October 31, 1936.)

471. Mr. D. H. Hyers: *Gateaux differential equations in Banach spaces.*

A. D. Michal and V. Elconin have established an existence theorem for the total differential equation $y(x;\ \xi)=F(x,\ y(x),\ \xi)$ in a Banach space, where F is linear in ξ and the left member is the Fréchet differential of the unknown function $y(x)$ (see *Completely integrable differential equations in abstract spaces*, Acta Mathematica (in press) or, Proceedings of the National Academy of Sciences, vol. 21 (1935), p. 534). The author proves an existence theorem for this equation in which F is merely additive and homogeneous but not necessarily continuous in ξ, and where the left member is interpreted as a Gateaux differential. (Received October 31, 1936.)

472. Mr. Victor Elconin: *The Wronskian of the functions* $x^{k-1}e^{\alpha_j x}$.

It is shown by three distinct methods that the Wronskian of the $n_1+n_2+\cdots+n_r$ functions $e^{\alpha_1 x}$, $xe^{\alpha_1 x}$, $\cdots$, $x^{n_1-1}e^{\alpha_1 x}$, $e^{\alpha_2 x}$, $\cdots$, $x^{n_r-1}e^{\alpha_r x}$, where the α_k are complex numbers, is

$$\prod_1^k{}_j(n_j-1)!!\prod_{r\geqq j>k\geqq 1}(\alpha_j-\alpha_k)^{n_j-n_k}e^{(n_1\alpha_1+n_2\alpha_2+\cdots+n_r\alpha_r)x}$$

where $m!!=1!2!\cdots m!$. (Received November 2, 1936.)

473. Mr. Victor Elconin: *Linear paraphrasing.*

E. T. Bell's fundamental theorem on arithmetical paraphrasing can be generalized and abstracted to describe all similar linear substitutions. Let S be a finite subset of a linear space, and σ any other finite set. S is *represented by* f_s *on* σ if the f_s for $s\epsilon\sigma$ are all the elements of S. $\rho\subset\sigma$ belongs to $R\subset S$ if R is represented by f_s on ρ. R is *irreducible* if $\sum_r^\rho\alpha_r f_r=0$ implies $\alpha_r=0$, *reducible* otherwise; the α_r, as in the following, are complex numbers, and the summation is for all $r\epsilon\rho$. R is *maximal* if it is irreducible and $R\subset U\subset S$, $R\neq U$, implies U is reducible. The basic theorem follows: *Suppose that* S, T *are finite subsets of linear spaces represented by* f_s, g_s *on* σ, *and that* ρ *belongs to a maximal subset of* S. *If* $f_s=\sum_r^\rho\alpha_{sr}f_r$ *implies* $g_s=\sum_r^\rho\alpha_{sr}b_r$, *then* $\sum_s^\sigma\alpha_s f_s=0$ *implies* $\sum_s^\sigma\alpha_s g_s=0$. A series of increasingly specialized sets S, T are considered and the characteristic property of ρ is determined in each case, for the last of which Bell's theorem (see Transactions of this Society, vol. 22 (1921), p. 5), generalized to arbitrary complex coefficients, is a corollary of the theorem above. (Received November 2, 1936.)

474. Mr. Victor Elconin: *Euler's integrability condition.*

Sarrus, De Morgan, Raabe, Joachimstal, and several others have given proofs none of which involves variational methods of Euler's theorem:

$F(x, y, dy/dx, \cdots, d^ny/dx^n)$ is an exact derivative if and only if $\partial F/\partial y - d/dx$ and $\partial F/\partial y_1 + \cdots + (-)^n d^n/dx^n$ and $\partial F/\partial y_n = 0$, where $y_k = d^ky/dx^k$. The inductive proof here presented appears to be the shortest elementary proof yet devised. (Received November 2, 1936.)

ERRATUM

On page 497 of this volume, abstract 42-7-311 (by Professor C. C. Camp), in the statement of the lemma, replace the words "otherwise zero" by "and $h-\theta$ when $h=h'+\theta$, $0<\theta<1$, where h' is an integer."