

THE CATEGORY AND BOREL CLASS OF CERTAIN
SUBSETS OF \mathcal{L}_p^*

BY J. C. OXTOBY

We consider the subsets of the Lebesgue spaces $\mathcal{L}_p([0, 1])$, ($p \geq 1$), corresponding to some familiar classes of functions, and show that they are Borel sets and of first category. The method consists in utilizing the boundedness properties of the functions to obtain representations of the sets in question in terms of closed sets, and in each case gives an evaluation of the Borel class.† By the set corresponding to a class is meant the set consisting of these and all equivalent functions.

LEMMA. *The class of functions f in \mathcal{L}_p such that $f(x) \leq g(x)$ for almost all x in $[a, b]$, ($0 \leq a < b \leq 1$), g being an arbitrary function in \mathcal{L}_p , constitutes a nowhere dense closed set in \mathcal{L}_p .*

Consider any sequence in the set, converging in \mathcal{L}_p to f . It is possible to pick a subsequence that converges pointwise to $f(x)$ almost everywhere.‡ It follows that $f(x) \leq g(x)$ almost everywhere in $[a, b]$, which shows that the set is closed. To prove that it is nowhere dense, it now suffices to show it is included in the derived set of its complement. Let f be any function in the set. Changing its value to $g(x) + 1$, say, on a small subinterval of $[a, b]$, we can obtain a function $f_\epsilon(x)$, in the complementary set, such that $\|f_\epsilon - f\| < \epsilon$. The lemma evidently remains true for the class of functions bounded below by $g(x)$ almost everywhere in $[a, b]$.

THEOREM 1. *The set C of points of \mathcal{L}_p corresponding to continuous functions is an $F_{\sigma\delta}$ set of first category.§*

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† For a general method of evaluating the Borel class of a set see Kuratowski, *Fundamenta Mathematicae*, vol. 17 (1931), pp. 249–272. For a specific result concerning the category of a set in \mathcal{L}_p , see E. S. Quade, this Bulletin, vol. 41 (1935), pp. 83–84.

‡ Hobson, *The Theory of Functions of a Real Variable*, vol. 2, second edition, 1926, p. 245.

§ An F_σ set is a denumerable union of closed sets. An $F_{\sigma\delta}$ set is a denumerable intersection of F_σ sets. A denumerable union of nowhere dense sets is said to be of first category.

Let (g_m, h_m) be an enumeration of all pairs of rational step-functions, g_m lower, h_m upper semi-continuous. Let E_{nm} be the class of all functions f in \mathcal{L}_p such that $g_m(x) \leq f(x) \leq h_m(x)$ almost everywhere in $[0, 1]$, and $0 \leq h_m(x) - g_m(x) \leq 1/n$ in $[0, 1]$. E_{nm} is closed and nowhere dense, since it is the intersection of two sets of the type considered in the lemma. We have the representation $C = \prod_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{nm}$. For, by the Heine-Borel theorem, any continuous function can be enclosed between two rational step-functions, conveniently taken semi-continuous, differing uniformly by less than $1/n$; and conversely, a function so enclosed almost everywhere is equivalent to a continuous function, the common limit of the bounding step-functions.

THEOREM 2. *The sets S_u and S_l of points of \mathcal{L}_p corresponding to upper and lower semi-continuous functions are $F_{\sigma\delta}$ sets of first category.*

Let $\{h_m\}$ be an enumeration of all upper semi-continuous rational step-functions. Denote by E_{nm} the set of functions f in \mathcal{L}_p such that $f(x) \leq h_m(x)$ almost everywhere in $[0, 1]$, and $\int_0^1 |f(x) - h_m(x)| dx \leq 1/n$. This is the intersection of a set of the type considered in the lemma with the set characterized by the second condition. To show that the latter is closed, consider any sequence $\{f_k\}$ in it converging in \mathcal{L}_p to f . Then f_k and f are in \mathcal{L}_1 , and, by the triangle and Hölder inequalities, we have

$$\begin{aligned} \int_0^1 |f - h_m| dx &\leq \int_0^1 |f_k - h_m| dx + \int_0^1 |f_k - f| dx \\ &\leq \frac{1}{n} + \left(\int_0^1 |f_k - f|^p dx \right)^{1/p}. \end{aligned}$$

Letting $k \rightarrow \infty$, we see that the set is closed. It follows that E_{nm} is closed and nowhere dense. We have $S_u = \prod_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{nm}$. For, by considering upper Darboux sums, an upper semi-continuous function is seen to be bounded above by step-functions, which may be taken rational, with integrals arbitrarily close to that of the function.* On the other hand, a function in \mathcal{L}_p that can

* Carathéodory, *Vorlesungen über reelle Funktionen*, second edition, 1927, p. 457. To apply this theorem it is sufficient that the function be bounded above. In our case, this follows from the upper semi-continuity, since we are considering the closed interval $[0, 1]$.

be so approximated almost everywhere is equivalent to the lower bound of the approximating sequence, an upper semi-continuous function. The theorem for S_l follows from the homeomorphism $Tf = -f$.

THEOREM 3. *The set R of points of \mathcal{L}_p corresponding to bounded Riemann integrable functions is the intersection of S_u and S_l and hence is an $F_{\sigma\delta}$ set of first category.*

Every bounded Riemann integrable function is in \mathcal{L}_p and continuous almost everywhere, hence equivalent to its maximum and minimum functions, respectively upper and lower semi-continuous.* On the other hand, two equivalent functions, respectively upper and lower semi-continuous, are continuous wherever equal, that is, almost everywhere. They are also bounded, hence Riemann integrable.

It is interesting to note that whereas the intersection of the classes of upper and lower semi-continuous functions is the class of continuous functions, nevertheless the intersection of the corresponding point sets in \mathcal{L}_p is not the set corresponding to continuous functions, but the larger set R .

THEOREM 4. *If $q > p$, then \mathcal{L}_q is contained in \mathcal{L}_p and is an F_σ set of first category in \mathcal{L}_p .†*

Denote by the symbol E_N the set of functions in \mathcal{L}_p such that $\int^1 |f(x)|^q dx \leq N$. Consider any sequence in E_N with limit in \mathcal{L}_p . We may suppose the sequence pointwise convergent almost everywhere. By truncating the functions to the bound m , applying the Lebesgue convergence theorem, and letting $m \rightarrow \infty$, the set E_N is seen to be closed. To show that it is also nowhere dense, consider any f in E_N , $\epsilon > 0$. Choose $\eta > 2\epsilon$ such that $\epsilon^p |\eta - 2N|^q > N\eta^p$. Then choose a subset \mathcal{E} of $[0, 1]$ of measure $(\epsilon/\eta)^p$ on which $|f(x)| \leq 2N$. This is possible, since for a function in E_N , $|f(x)| \leq 2N$ on a set of measure at least $1/2$, and

* Hobson, op. cit., vol. 1, third edition, 1927, pp. 465, 308.

† This theorem, at least as regards category, was proved earlier by Garrett Birkhoff and E. S. Quade, but was not published. The evaluation of the Borel class as F_σ is here the precise determination, since the set is not closed, and neither is it G_δ . See Mazur and Sternbach, *Studia Mathematica*, vol. 4 (1934), p. 50, where it is shown that a linear G_δ set is necessarily closed.

η was so chosen that $(\epsilon/\eta)^p < 1/2$. Define $f_\epsilon(x) = f(x) + \eta$ on \mathcal{E} and equal to $f(x)$ elsewhere. Then $\|f_\epsilon - f\| = \epsilon$, while

$$\int_0^1 |f_\epsilon(x)|^q dx > N.$$

Since $\mathcal{L}_q = \sum_{N=1}^{\infty} E_N$, the theorem is established.*

THEOREM 5. *The class of functions in \mathcal{L}_p bounded away from some value almost everywhere in a neighborhood of some point of $[0, 1]$ constitutes an F_σ set of first category.*

Here we let $E(r_1, r_2, r_3, r_4)$ be the nowhere dense closed set consisting of all functions f in \mathcal{L}_p such that $f(x) \leq r_3$ or $f(x) \geq r_4$, ($r_3 < r_4$), for almost all x in $r_1 \leq x \leq r_2$, ($0 \leq r_1 < r_2 \leq 1$). The set in question is obtained by summing over all rational r_1, r_2, r_3, r_4 satisfying the stated inequalities.

The methods used in proving the above theorems can be adapted to treat questions of relative category. For example, we have the following theorem.

THEOREM 6. *Relative to the complete subspace $B_{M,p}$ consisting of all functions f in \mathcal{L}_p such that $|f(x)| \leq M$, ($0 \leq x \leq 1$; $M > 0$), the sets $CB_{M,p}$, $S_u B_{M,p}$, $S_l B_{M,p}$, $RB_{M,p}$, are F_σ of first category. Except for a set of first category, every function in $B_{M,p}$ has upper and lower Riemann integrals equal to M and $-M$, respectively.*

The proof consists in showing that the set of functions in $B_{M,p}$ bounded away from M or $-M$ almost everywhere in some subinterval forms a set of first category in $B_{M,p}$, and in noting that a function in any one of the sets mentioned in the theorem either has this property or is equivalent to one of the constant functions M , $-M$.

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* The latter half of this proof can be avoided by appealing to a theorem of Banach, *Théorie des Opérations Linéaires*, 1932, p. 36, according to which a proper linear subset of \mathcal{L}_p is necessarily of first category if it is a Borel set. This remark applies also to the sets C and R .