# GENERAL TENSOR ANALYSIS*

BY A. D. MICHAL

1. *Introduction*. The *coordinates* of the traditional tensor analysis are points in a finite dimensional arithmetic space while those of the author's infinitely dimensional tensor analysis are points (functions) in a function space. The purpose of the present paper is to present the elements of a general tensor analysis that will include as instances the tensor calculi just mentioned. This general point of view has already been instrumental in the more elegant and further development of the function space instances.

For a space of *coordinates* we take a Banach space. The geometrical objects studied are contravariant vector fields, linear connections, multilinear forms and their covariant differentials. Non-holonomic geometric objects are not considered here, as I intend to pursue their study elsewhere.

2. *Abstract Coordinate Transformations*. We consider a Hausdorff† topological space $T$ whose neighborhoods are mapped homeomorphically on an open set $S$ of a Banach space $E$ by mapping functions called *coordinate systems*.‡ If two neighborhoods intersect we have two mappings of their intersection on open subsets $S_1$ and $S_2$ of $S$. This establishes a homeomorphism $\bar{x}(x)$, called a *coordinate transformation*, that takes an open set $S_1 \subset S$ into an open set $S_2 \subset S$. As further restrictions we demand that the Fréchet differentials $\bar{x}(x;\, \delta x)$ and $x(\bar{x};\, \delta\bar{x})$ of $\bar{x}(x)$ and its inverse $x(\bar{x})$ exist in $S_1$ and $S_2$, respectively. Finally, to deal with the law of transformation of a linear connection and with covariant differentials, we shall assume that the second Fréchet differential $\bar{x}(x;\, \delta_1 x;\, \delta_2 x)$ exists in $S_1$ continuously in $x$.

* Presented to the Society, April 11, 1936.

† For the purposes of the present paper alone it is merely necessary to postulate that $T$ is a Fréchet neighborhood space and that the coordinate systems are reciprocal (1-1) transformations.

‡ All the results of the paper continue to hold if we take the *coordinate systems* to be homeomorphisms of Hausdorff neighborhoods onto open subsets $\Sigma \subset S$, where $S$ is a fixed open set in $E$ and is itself a homeomorphic map of some Hausdorff neighborhood.

Since $\bar{x}(x)$ and $x(\bar{x})$ are mutually inverse, we see that $\bar{x}(x;\ \delta x)$ is a solvable linear function* of $\delta x$ with $x(\bar{x};\ \delta\bar{x})$ as inverse. The symmetry of $\bar{x}(x;\ \delta_1 x;\ \delta_2 x)$ in $\delta_1 x$ and $\delta_2 x$ follows from a theorem proved by Kerner.† An application of a theorem‡ of Michal and Elconin on solvable linear functions depending on a parameter shows that the second Fréchet differential $x(\bar{x};\ \delta_1\bar{x};\ \delta_2\bar{x})$ exists and is continuous in $\bar{x}$ throughout $S_2$. On differentiating the evident identity $\bar{x}(x;\ x(\bar{x};\ \delta_1\bar{x})) = \delta_1\bar{x}$ we obtain the useful identity

$$(1) \qquad \bar{x}(x;\ \delta_1 x;\ \delta_2 x) + \bar{x}(x;\ x(\bar{x};\ \delta_1\bar{x};\ \delta_2\bar{x})) = 0.$$

The validity of this relation follows on making special use of a theorem§ on the *total* differential of a linear function that depends on a parameter. In the sequel we shall refer to this theorem as Theorem M.

3. *Covariant Differential of a Contravariant Vector Field and the Transformation Law of a Linear Connection.* Let $P_0$ be any chosen point of the Hausdorff space $T$. A geometric object whose abstract components undergo the transformation $\bar{\xi} = \bar{x}(x(P_0);\ \xi)$ will be called a *contravariant vector* (associated with $P_0$). To avoid long circumlocutions we shall say that $\xi$ *is* a contravariant vector. Similar abbreviations will be made for the components of other geometric objects. That a differential $\delta x$ is a contravariant vector is clear from the evident formulas

$$\delta\bar{x} = \bar{x}(x;\ \delta x), \qquad \delta x = x(\bar{x};\ \delta\bar{x}).$$

A *contravariant vector field* (c.v.f. for brevity) is a geometric object with abstract components. More precisely, to every Hausdorff neighborhood with coordinate system $x(P)$ there exists a function (called component) $\xi(x)$ on $S$ to $E$ such that in the intersection of two Hausdorff neighborhoods $\bar{\xi}(\bar{x}) = \bar{x}(x;\ \xi(x))$ under a transformation of coordinates $\bar{x} = \bar{x}(x)$.

* A. D. Michal and V. Elconin, *Differential properties of abstract transformation groups with abstract parameters*, American Journal of Mathematics, vol. 59 (1937), pp. 129–143.

† M. Kerner, Annals of Mathematics, (2), vol. 34 (1933), pp. 546–572. See also A. D. Michal and V. Elconin, *Completely integrable differential equations in abstract spaces* (accepted for publication in the Acta Mathematica).

‡ A. D. Michal and V. Elconin, Acta Mathematica, loc. cit.

§ A. D. Michal, *Postulates for a linear connection*, Annali di Matematica, (1936), pp. 197–220.

Let $\xi_1$ and $\xi_2$ be two arbitrary contravariant vectors. The object whose components $\Gamma(x, \xi_1, \xi_2)$ and $\bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2)$ are bilinear functions of the vectors will be called a *linear connection** if, in the intersection of two Hausdorff neighborhoods, the components have the law of transformation

$$(2) \qquad \bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) + \bar{x}(x; x(\bar{x}; \bar{\xi}_1; \bar{\xi}_2)).$$

With the aid of relation (1) we can immediately rewrite (2) in the equivalent form

$$(3) \qquad \bar{\Gamma}(\bar{x}, \bar{\xi}_1, \bar{\xi}_2) = \bar{x}(x; \Gamma(x, \xi_1, \xi_2)) - \bar{x}(x; \xi_1; \xi_2).$$

THEOREM 1. *Let the bilinear function $\Gamma(x, \xi_1, \xi_2)$ in the contravariant vectors $\xi_1$, $\xi_2$ be the component in the $x(P)$ coordinate system of an object with components. Then, a necessary and sufficient condition that*†

$$(4) \qquad \delta\xi(x) + \Gamma(x, \xi(x), \delta x)$$

*be the component in the $x(P)$ coordinate system of a contravariant vector field for every Fréchet differentiable contravariant vector field $\xi(x)$ is that $\Gamma(x, \xi_1, \xi_2)$ be the component of a linear connection.*

PROOF. To prove the necessity of the condition we have by hypothesis

$$(5) \qquad \delta\bar{\xi}(\bar{x}) + \bar{\Gamma}(\bar{x}, \bar{\xi}(\bar{x}), \delta\bar{x}) = \bar{x}(x; \delta\xi(x) + \Gamma(x, \xi(x), \delta x)).$$

But, since $\xi(x)$ is a differentiable contravariant vector field and the second differential $\bar{x}(x; \delta_1 x; \delta_2 x)$ exists, it follows that $\delta\bar{\xi}(\bar{x})$ exists and is given by

$$(6) \qquad \delta\bar{\xi}(\bar{x}) = \bar{x}(x; \xi(x); \delta x) + \bar{x}(x; \delta\xi(x)).$$

* Further generalizations are possible. One can consider still another Banach space $E_1$ and another type of contravariant vector field with component $V(x)$, a function on $E$ to $E_1$, subject to a law of transformation $\bar{V}(\bar{x}) = M(x, V(x))$, where $M(x, y)$ is a solvable function of $y$. The covariant differentials of multilinear forms $F(x, \xi_1, \xi_2, \cdots, \xi_n, V_1, V_2, \cdots, V_i)$ can then be developed with the aid of $\Gamma(x, \xi, \delta x)$ and another kind of linear connection $K(x, V, \delta x)$, a function as $EE_1E$ to $E_1$, whose law of transformation is

$$\bar{K}(\bar{x}, \bar{V}, \delta\bar{x}) = M(x, K(x, V, \delta x)) - M(x, V; \delta x).$$

† We shall often write the Fréchet differential $f(x; \delta_i x)$ as $\delta_i f(x)$.

We thus obtain immediately the law of transformation

$$(7)\qquad \overline{\Gamma}(\bar{x}, \bar{\xi}(\bar{x}), \delta\bar{x}) = \bar{x}(x; \Gamma(x, \xi(x), \delta x)) - \bar{x}(x; \xi(x); \delta x).$$

The sufficiency of the condition follows by a reversal of the steps in the obvious manner.

DEFINITION. *If* $\xi(x)$ *is a contravariant vector field and* $\Gamma(x, \xi, \delta x)$ *is a linear connection, then the contravariant vector field* (4) *will be called the covariant differential of* $\xi(x)$ *and will be denoted by* $\xi(x|\,\delta x)$.

4. *Covariant Differential of Multilinear Forms in Contravariant Vector Fields.* Our treatment of successive covariant differentials will be made to depend on the results of the following fundamental theorem.

THEOREM 2. *Let* $\Gamma(x, \xi, \delta x)$ *be a linear connection and* $F(x, \xi_1, \xi_2, \cdots, \xi_n)$ *a function with the following properties:*

(i) $F$ *is a* c.v.f. *valued multilinear form in the* $n$ *arbitrary contravariant vectors* $\xi_1, \cdots, \xi_n$;

(ii) *the partial Fréchet differential* $F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x)$ *exists and is continuous in* $x$.

*Then the function* $F(x, \xi_1, \xi_2, \cdots, \xi_n|\,\delta x)$ *defined by the equation*

$$(8)\qquad \begin{aligned} F(x, \xi_1, \xi_2, \cdots, \xi_n \,|\, \delta x) &= F(x, \xi_1, \xi_2, \cdots, \xi_n; \delta x) \\ &- \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n) \\ &+ \Gamma(x, F(x, \xi_1, \xi_2, \cdots, \xi_n), \delta x) \end{aligned}$$

*is a* c.v.f. *valued multilinear form in* $\xi_1, \xi_2, \cdots, \xi_n, \delta x$. *We shall call* $F(x, \xi_1, \cdots, \xi_n|\,\delta x)$ *the covariant differential of* $F(x, \xi_1, \cdots, \xi_n)$.

PROOF. We shall give here the details of proof only for the case $n=1$ as the method of proof for the general case differs in no essential manner from that of this special case. By hypothesis, then, we have

$$(9)\qquad \overline{F}(\bar{x}, \bar{\xi}) = \bar{x}(x; F(x, \xi)),$$

from which we obtain

$$(10)\qquad \overline{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) = \phi(x, \xi; \delta x) - \overline{F}(\bar{x}, \delta\bar{\xi}),$$

where $\phi(x, \xi) = \bar{x}(x; F(x, \xi))$. On using $\bar{\xi} = \bar{x}(x; \xi)$, we find

$$(11) \qquad \bar{F}(\bar{x}, \delta\bar{\xi}) = \bar{F}(\bar{x}, \bar{x}(x; \xi; \delta x)).$$

Hence (10) immediately reduces to

$$(12) \qquad \bar{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) = \phi(x, \xi; \delta x) - \bar{F}(\bar{x}, \bar{x}(x; \xi; \delta x)).$$

From the laws of transformation (3) and (9), we evidently have

$$(13) \quad \bar{F}(\bar{x}, \bar{\Gamma}(\bar{x}, \bar{\xi}, \delta\bar{x})) = \bar{x}(x; F(x, \Gamma(x, \xi, \delta x))) - \bar{F}(\bar{x}, \bar{x}(x; \xi; \delta x)).$$

From (3) we obtain also

$$(14) \quad \bar{\Gamma}(\bar{x}, \bar{F}(\bar{x}, \bar{\xi}), \delta\bar{x}) = \bar{x}(x; \Gamma(x, F(x, \xi), \delta x)) - \bar{x}(x; F(x, \xi); \delta x).$$

With the aid of (12), (13), (14), and a special use of Theorem M on Fréchet differentials, we finally obtain the law of transformation

$$(15) \qquad \bar{F}(\bar{x}, \bar{\xi} \mid \delta\bar{x}) = \bar{x}(x; F(x, \xi \mid \delta x)),$$

where

$$(16) \qquad \begin{aligned} F(x, \xi \mid \delta x) &= F(x, \xi; \delta x) + \Gamma(x, F(x, \xi), \delta x) \\ &\quad - F(x, \Gamma(x, \xi, \delta x)), \end{aligned}$$

and

$$\bar{F}(\bar{x}, \bar{\xi} \mid \delta\bar{x}) = \bar{F}(\bar{x}, \bar{\xi}; \delta\bar{x}) + \bar{\Gamma}(\bar{x}, \bar{F}(\bar{x}, \bar{\xi}), \delta\bar{x}) - \bar{F}(\bar{x}, \bar{\Gamma}(\bar{x}, \bar{\xi}, \delta\bar{x})).$$

By the Gateaux limit method of evaluating a Fréchet differential and by a theorem of Banach on the limit of a convergent sequence of linear functions, it can readily be shown* that $F(x, \xi; \delta x)$ is a bilinear function of $\xi$ and $\delta x$. Hence the covariant differential $F(x, \xi \mid \delta x)$ is bilinear in $\xi$ and $\delta x$.

Since the covariant differential of a c.v.f. valued multilinear form in $n$ contravariant vectors is itself a c.v.f. valued multilinear form in $n+1$ contravariant vectors, we may apply Theorem 2 repeatedly and obtain higher order covariant differentials $F(x, \xi_1, \cdots, \xi_n \mid \delta_1 x \mid \cdots \mid \delta_r x)$ of the original multilinear form $F(x, \xi_1, \cdots, \xi_n)$.

* A. D. Michal, Annali di Matematica, loc. cit.

Let $\xi(x)$ be a contravariant vector field with a continuous second differential $\xi(x;\,\delta_1 x;\,\delta_2 x)$ in $x$ and let the linear connection $\Gamma$ possess a continuous partial differential $\Gamma(x,\,\xi_1,\,\xi_2;\,\delta x)$ in $x$. If we further assume that the coordinate transformations $\bar{x}(x)$ have continuous third differentials $\bar{x}(x;\,\delta_1 x;\,\delta_2 x;\,\delta_3 x)$ in $x$, then it can be shown that $x(\bar{x};\,\delta_1\bar{x};\,\delta_2\bar{x};\,\delta_3\bar{x})$ exists and is continuous in $\bar{x}$ and that the above restrictions on $\xi(x)$ and on the linear connection are invariant under such transformations of coordinates. If we then make special use of Theorem M, we find that

$$\begin{aligned}\xi(x\,|\,\delta_1 x\,|\,\delta_2 x) - \xi(x\,|\,\delta_2 x\,|\,\delta_1 x) &= B(x,\,\xi(x),\,\delta_1 x,\,\delta_2 x)\\ &\quad - 2\xi(x\,|\,\Omega(x,\,\delta_1 x,\,\delta_2 x)),\end{aligned}\tag{17}$$

where $B(x,\,\xi,\,\delta_1 x,\,\delta_2 x)$, called the *curvature form*, is defined by

$$\begin{aligned}B(x,\,\xi,\,\delta_1 x,\,\delta_2 x) &= \Gamma(x,\,\xi,\,\delta_1 x;\,\delta_2 x) - \Gamma(x,\,\xi,\,\delta_2 x;\,\delta_1 x)\\ &\quad + \Gamma(x,\,\Gamma(x,\,\xi,\,\delta_1 x),\,\delta_2 x) - \Gamma(x,\,\Gamma(x,\,\xi,\,\delta_2 x),\,\delta_1 x),\end{aligned}\tag{18}$$

and where $\Omega(x,\,\delta_1 x,\,\delta_2 x)$, called the *torsion form*, is defined by

$$\Omega(x,\,\delta_1 x,\,\delta_2 x) = \frac{1}{2}\left\{\Gamma(x,\,\delta_1 x,\,\delta_2 x) - \Gamma(x,\,\delta_2 x,\,\delta_1 x)\right\}.\tag{19}$$

It follows from (3) and Kerner's theorem on symmetry of continuous second differentials that the torsion form $\Omega(x,\,\delta_1 x,\,\delta_2 x)$ is a c.v.f. valued bilinear form. Hence with the aid of Theorem 1, Theorem 2, and (17), we see that the curvature form $B(x,\,\xi,\,\delta_1 x,\,\delta_2 x)$ is a c.v.f. valued trilinear form. We can thus form the sequence of covariant differentials of the curvature form $B(x,\,\xi_1,\,\xi_2,\,\xi_3)$:

$$B(x,\,\xi_1,\,\xi_2,\,\xi_3\,|\,\delta_1 x\,|\,\cdots\,|\,\delta_r x), \qquad (r = 1,\,2,\,\cdots,\,n).$$

In order that this sequence of covariant differentials exist it is sufficient to assume the existence of $\Gamma(x,\,\xi_1,\,\xi_2;\,\delta_1 x;\,\cdots;\,\delta_{n+1}x)$ and its continuity in $x$. These restrictions will be invariant under coordinate transformations if we assume that the coordinate transformations $\bar{x}(x)$ possess a continuous differential of order $n+3$ in $x$.

By a slight variation of the method of proof for Theorem 2, the following theorem can be proved.

THEOREM 3. *If in the hypotheses of Theorem* 2 *we replace* (i) *by the following*:

(i′) *$F$ is an absolute scalar multilinear form in the $n$ arbitrary contravariant vectors* $\xi_1, \cdots, \xi_n$,

*then the function* $F(x, \xi_1, \cdots, \xi_n \mid \delta x)$ *defined by*

$$F(x, \xi_1, \cdots, \xi_n \mid \delta x) = F(x, \xi_1, \cdots, \xi_n; \delta x) - \sum_{i=1}^{n} F(x, \xi_1, \cdots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \cdots, \xi_n) \tag{20}$$

*is an absolute scalar multilinear form in* $\xi_1, \cdots, \xi_n, \delta x$. *We shall then call* $F(x, \xi_1, \cdots, \xi_n \mid \delta x)$ *the covariant differential of* $F(x, \xi_1, \cdots, \xi_n)$.

The conclusion of this theorem continues to hold if the numerical valued form $F$ is replaced by a form $F$ with values in a Banach space.

5. *The "Stokean" Covariant Differential of Alternating Forms.* In this section we do not assume the existence of a linear connection. It is possible nevertheless to obtain with Fréchet differentiation an absolute alternating form of one higher order from a given absolute alternating form in contravariant vectors.

THEOREM 4. *Let* $\omega(x, \xi_1, \cdots, \xi_n)$ *be a function with the following properties*:

(i) *$\omega$ is an absolute scalar multilinear form in the contravariant vectors* $\xi_1, \cdots, \xi_n$;

(ii) *$\omega$ is alternating in* $\xi_1, \cdots, \xi_n$ *(that is, completely skew-symmetric)*;

(iii) *the second partial differential* $\omega(x, \xi_1, \cdots, \xi_n; \delta_1 x; \delta_2 x)$ *exists and is continuous in $x$.*

*Then under the transformations of coordinates of* §3:

($C_1$) *the function* $\omega(x, \xi_1, \cdots, \xi_n : \delta x)$, *called the Stokean covariant differential of* $\omega(x, \xi_1, \cdots, \xi_n)$, *defined by*

$$\omega(x, \xi_1, \cdots, \xi_n : \delta x) = \omega(x, \xi_1, \cdots, \xi_n; \delta x) - \sum_{i=1}^{n} \omega(x, \xi_1, \cdots, \xi_{i-1}, \delta x, \xi_{i+1}, \cdots, \xi_n; \xi_i), \tag{21}$$

*is an alternating absolute scalar multilinear form in* $\xi_1, \cdots, \xi_n, \delta x$;

$$\omega(x, \xi_1, \cdots, \xi_n : \delta_1 x : \delta_2 x) = 0. \tag{$C_2$}$$

It is clear from this theorem that the Cartan-Goursat calculus of alternating forms* can be developed in Hausdorff spaces with Banach coordinates.

In conclusion we note that Theorem 4 continues to hold if the numerically valued form $\omega$ is replaced by a form $\omega$ with values in a Banach space.

CALIFORNIA INSTITUTE OF TECHNOLOGY

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# ON A THEOREM OF ENGEL†

BY MAX ZORN

1. *Introduction.* The theorem of Engel which we intend to study in this paper deals with Lie algebras where an identity $(a(a(a \cdots (ab)) \cdots)) = 0$ holds for arbitrary elements $a$ and $b$. Under various assumptions it has been shown that in this case all products with sufficiently many factors vanish.

The latest result in this direction was a proof,‡ found first by van Kampen, which holds for finite Lie algebras over any field of characterstic zero. The method is *rational*, but it involves the theory of associative algebras and the theory of traces. Another proof of equal generality, with less accent on the theory of traces, has recently been sketched by the writer.§

The new proof to be offered in the present paper dispenses with every apparatus of matrices, traces, and associative systems. It does not presuppose any knowledge about Lie systems. The material advantage of our direct method is the fact that no reference field is required, and that the question of characteristics never enters the discussion.

2. *Definitions.* DEFINITION 1. *A system L of elements a, b,* $\cdots$ *is called a Lie ring (with respect to a commutative ring* P *of*

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* E. Goursat, *Leçons sur le Problème de Pfaff*, 1922; E. Cartan, *Leçons sur les Invariants Intégraux*, 1922; E. Kahler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, 1934.

† Substituted for another paper, which was presented to the Society, June 18, 1936. See the last footnote on this page.

‡ See N. Jacobson, *Rational methods in the theory of Lie algebras*, Annals of Mathematics, vol. 36, p. 875.

§ See this Bulletin, Abstract 42-7-266. (Erroneously the theorem in question is there attributed to Lie.)