ON A CLASS OF RECURRENT SEQUENCES*

BY H. E. ROBBINS

The first example of a recurrent, non-periodic sequence was given by Marston Morse† in connection with the geodesics on a surface of negative curvature. A discussion of their significance, together with a general method of definition may be found in Birkhoff's *Dynamical Systems*, 1927, p. 246. In this note they will be considered independently of their origin, and a class of such sequences with certain interesting properties will be defined. (Theorem 1 of the present paper, except insofar as it refers to the particular sequence under discussion, is therefore not new.) As a preliminary to this we shall make certain definitions.

A *sequence* is a doubly-infinite row of the symbols 1 and 2: $\cdots c_{-3}c_{-2}c_{-1}c_0c_1c_2c_3\cdots$, where each c_i is either a 1 or a 2. A *block* is a set of consecutive members of a sequence: $c_mc_{m+1}\cdots c_n$ and has length s if there are s symbols in it. A sequence is *periodic* if there exists an integer p such that $c_{n+p}=c_n$ for all n. A sequence is *recurrent* if there exists a function of integers $f(n)$ with integral values, such that any block of length n chosen anywhere in the sequence is contained as a block in any block of length $f(n)$. The least such function $f(n)$ will be called the *ergodic function* of the sequence.

As an example of such a sequence, let

$$a_0 = 12, \quad a_1 = a_0^{-1}a_0a_0 = 21\underline{12}12, \quad \cdots, \quad a_{n+1} = a_n^{-1}a_na_n, \quad \cdots.$$

To define our sequence we number the symbols starting with the original

$$\begin{array}{c} \cdots c_0c_1 \cdots \\ \cdots 12 \cdots \end{array}$$

which has been underlined above. We may omit a more precise definition of the nth symbol since it will be unnecessary for our present purpose. We note that

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† Transactions of this Society, vol. 22 (1921), p. 94.

$$(a_n^{-1}a_na_n)^{-1} = a_n^{-1}a_n^{-1}a_n,$$

where, as above, a_n^{-1} denotes the symbols of a_n in reverse order.

THEOREM 1. *The sequence thus defined is recurrent but not periodic.*

PROOF OF RECURRENCE. First of all we note that there are $2\cdot 3^n$ symbols in the nth stage of the construction, a_n. Now let us choose any block of length s:

$$c_{m+1}c_{m+2}\cdots c_{m+s}, \tag{1}$$

and let n be the smallest integer such that $2\cdot 3^n \geqq s$.

Since for any m the sequence may be regarded as built up of a succession of blocks a_m and a_m^{-1} in some order, it follows that the block (1) must be contained in one of the following blocks:

$$a_na_n, \qquad a_na_n^{-1}, \qquad a_n^{-1}a_n, \qquad a_n^{-1}a_n^{-1}. \tag{2}$$

But

$$a_{n+1} = a_n^{-1}a_na_n, \qquad a_{n+1}^{-1} = a_n^{-1}a_n^{-1}a_n,$$

$$a_{n+2} = a_{n+1}^{-1}a_{n+1}a_{n+1} = a_n^{-1}a_n^{-1}a_na_n^{-1}a_na_na_n^{-1}a_na_n,$$

$$a_{n+2}^{-1} = a_n^{-1}a_n^{-1}a_na_n^{-1}a_n^{-1}a_na_n^{-1}a_na_n,$$

and *each* of the blocks (2) is contained in *both* a_{n+2} and a_{n+2}^{-1}. It follows that the block (1) is contained in *any* block of length twice the length of a_{n+2}, that is, of length $4\cdot 3^{n+2}$. So if we set $f(s)=4\cdot 3^{n+2}$, the sequence is recurrent with ergodic function at most equal to $f(s)$. To express $f(s)$ explicitly in terms of s, or rather, to get an upper bound for it, we proceed as follows. By definition, n is the smallest integer such that $2\cdot 3^n \geqq s$, that is, such that $n \geqq \log_3(s/2)$. Therefore, $n < \log_3(s/2)+1$. It follows that

$$f(s) \leqq 4\cdot 3^{n+2} < 4\cdot 3^{\log_3(s/2)+3} = 4\cdot\frac{s}{2}\cdot 27 = 54\cdot s,$$

that is, $f(s) < 54\cdot s$.*

PROOF OF NON-PERIODICITY. It is easily seen that periodic sequences have ergodic functions of the form $f(n)=n+c$, where

* By a more detailed analysis of the sequence, a much lower bound can be found for the ergodic function $f(s)$.

c is some constant. We shall show that the ergodic function of our sequence is not of this form. The proof will be based on the following lemma.

LEMMA 1. *For all* $n=1, 2, 3, \cdots$ *the block* a_n^{-1} *is not contained in* a_na_n *or in* $a_na_n^{-1}$ *or* $a_n^{-1}a_n$, *except as the initial or final block in the latter two.*

PROOF.

$$a_1a_1 = 211212211212,$$
$$a_1^{-1}a_1 = 212112211212, \qquad a_1^{-1} = 212112,$$
$$a_1a_1^{-1} = 211212212112,$$

and the assertion is obvious on inspection. Now suppose the assertion true for $n=m$. By definition

$$a_{m+1} = a_m^{-1}a_ma_m, \qquad a_{m+1}^{-1} = a_m^{-1}a_m^{-1}a_m,$$
$$a_{m+1}a_{m+1} = a_m^{-1}a_ma_ma_m^{-1}a_ma_m,$$
$$a_{m+1}^{-1}a_{m+1} = a_m^{-1}a_m^{-1}a_ma_m^{-1}a_ma_m,$$
$$a_{m+1}a_{m+1}^{-1} = a_m^{-1}a_ma_ma_m^{-1}a_m^{-1}a_m,$$

and inspection again shows the truth of the assertion for $n=m+1$. Now to show non-periodicity, we choose some $n=1, 2, 3, \cdots$ such that $2\cdot 3^n>c$, where c is an arbitrary positive integer. The blocks a_n^{-1} and a_na_n both occur in the sequence, but the length of $a_n^{-1}=2\cdot 3^n$, the length of $a_na_n=4\cdot 3^n>2\cdot 3^n+c$, while a_na_n contains no sub-block of the form a_n^{-1} by the lemma. Thus our sequence cannot be periodic. This concludes the proof of Theorem 1. We may state explicitly the following corollary.

COROLLARY. *There exist recurrent, non-periodic sequences whose ergodic functions are bounded above by linear functions (without constant terms).*

Having obtained a recurrent, non-periodic sequence whose ergodic function is bounded above by a linear function and which is thus, in a sense, close to the periodic case, we turn our attention to the other extreme and ask whether there exist r. n. p. sequences whose ergodic functions increase arbitrarily rapidly. We may state the problem as follows:

Given a function $R(n)$ (of integers, with integral values) such that

$$R(n) > n, \tag{3}$$

does there exist an r. n. p. sequence with ergodic function $f(n)$ such that for each n greater than some integer d, $f(n) > R(n)$; that is, does there exist a block of length $\leqq n$ and a block of length $> R(n)$ which does not contain it as a sub-block?

We shall see that this question is to be answered in the affirmative. Let us define a sequence as follows:

$$a_0 = 12, \qquad a_1 = a_0^{-1}\underbrace{a_0 a_0 \cdots a_0}_{R(6)\text{ times}}, \cdots, a_{n+1} = a_n^{-1}\underbrace{a_n a_n \cdots a_n}_{R[3\cdot l(a_n)]\text{ times}},$$

where $l(a_n)$ denotes the number of symbols in a_n. The precise method we choose to number the symbols of the sequence is of course arbitrary, but we may start with the "original" $\cdots$ 12 $\cdots$ as in the previous case. That this sequence is r. n. p. follows by much the same reasoning as in the previous simpler case. We shall confine our attention to proving that the ergodic function increases rapidly enough. The proof will be based on the following lemma.

LEMMA 2. *For any $n \geqq 1$, the block $a_n a_n \cdots a_n$ of arbitrary length contains no sub-block of the form a_n^{-1}.*

PROOF. It will be sufficient to show that this is true for the block $a_n a_n$. The assertion holds for $n = 1$:

$$a_1 = a_0^{-1}\underbrace{a_0 a_0 \cdots a_0} = 21\ \underbrace{12\ 12 \cdots 12},$$

$$a_1 a_1 = 21\ \underbrace{12\ 12 \cdots 12}\ 21\ \underbrace{12\ 12 \cdots 12},$$

$$a_1^{-1} = \underbrace{21\ 21 \cdots 21}\ 12.$$

Now assume it to hold for $n = m$. By definition,

$$a_{m+1}a_{m+1} = a_m{}^1\underbrace{a_m \cdots a_m}_{R[3\cdot l(a_m)]} a_m^{-1}\underbrace{a_m \cdots a_m}_{R[3\cdot l(a_m)]},$$

$$a_{m+1}^{-1} = \underbrace{a_m^{-1}a_m^{-1} \cdots a_m^{-1}}_{R[3\cdot l(a_m)]} a_m,$$

and from (2) the assertion is seen to hold for $n=m+1$, proving the lemma.

Now to prove the theorem, we choose an arbitrary integer s, where $l(a_n) \leqq s \leqq l(a_{n+1})$, $(n=1, 2, 3, \cdots)$, and suppose that $p \cdot l(a_n) \leqq s \leqq (p+1) \cdot l(a_n)$, where $1 \leqq p \leqq R[3 \cdot l(a_n)]$. We observe that

$$a_{n+1} = a_n^{-1} \underbrace{a_n \cdots a_n}_{R[3 \cdot l(a_n)] \text{ times}},$$

$$a_{n+2} = a_{n+1}^{-1} \underbrace{a_{n+1} \cdots a_{n+1}}_{R[3 \cdot l(a_{n+1})] \text{ times}},$$

$$a_{n+2}^{-1} = \underbrace{[a_n^{-1} \cdots a_n^{-1} a_n][a_n^{-1} \cdots a_n^{-1} a_n] \cdots [a_n^{-1} \cdots a_n^{-1} a_n]}_{R[3 \cdot l(a_{n+1})] \text{ blocks}} \cdots .$$

The number of symbols in this initial portion of a_{n+2}^{-1} is equal to $R[3 \cdot l(a_{n+1})] \cdot [R(3 \cdot l(a_n))+1] \cdot l(a_n)$, while it contains no sub-block of the form $a_n a_n a_n$ by Lemma 2. Thus, if $p \geqq 3$, the theorem is proved. If $p<3$, that is, $l(a_n) \leqq s \leqq 3 \cdot l(a_n)$, we simply observe that the block $a_n a_n \cdots a_n$ which is contained in a_{n+1} contains no sub-block of the form a_n^{-1} by the lemma. But its length is $>R[3 \cdot l(a_n)]$, which completes the proof of Theorem 2.

THEOREM 2. *There exist recurrent, non-periodic sequences whose ergodic functions increase arbitrarily rapidly.*

HARVARD UNIVERSITY