

**THE GEOMETRY OF THE WHIRL-MOTION GROUP G_6 :
ELEMENTARY INVARIANTS***

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In this paper we shall study the elementary geometry of the oriented lineal elements of the plane with respect to the whirl-motion group G_6 . We give results in addition to those found in a paper by Kasner, *The group of turns and slides and the geometry of turbines*, published in 1911 in the American Journal of Mathematics (vol. 33, pp. 193–202) and the paper by the authors, *The geometry of turbines, flat fields, and differential equations*, published in 1937 in the American Journal of Mathematics (vol. 59, pp. 545–563). The present paper can be read independently of the two earlier papers and contains the foundation of a new geometry of differential elements.

We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A turn T_α converts each element into one having the same point and a direction making a fixed angle α with the original direction. By a slide S_k the line of the element remains the same and the point moves along the line a fixed distance k . These transformations together generate a continuous group of three parameters which we call the group of *whirl* transformations. It is noted that the group of whirls is isomorphic to the group of motions. These two three-parameter groups are commutative and together generate a new group of six parameters which we term the *whirl-motion group* G_6 . It is our purpose to study the simple geometry of this group G_6 . We find the fundamental invariants.

We define ∞^1 elements to be a *series* of elements; this includes a union or curve as a special case. A set of ∞^2 elements is said to form a *field* of elements, which corresponds to a differential equation of the first order $F(x, y, y') = 0$. A *turbine* is the series which is obtained by applying a turn T_α to the elements of an oriented circle (which may be a null circle). A *flat field* is the field that is obtained from the totality of all elements which are determined by the set of all oriented circles containing a given element. We obtain the elementary relationships between elements, turbines, and flat fields.

For the analytic representation, it will be convenient to define an element by the coordinates (u, v, w) where v is the perpendicular from the origin, u is the angle between the perpendicular and the initial

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line, and w is the distance between the foot of the perpendicular and the point of the element.

1. **The whirl-motion group G_6 of transformations.*** The equations of the turn T_α are

$$(1) \quad \tilde{u} = u + \alpha, \tilde{v} = v \cos \alpha + w \sin \alpha, \tilde{w} = -v \sin \alpha + w \cos \alpha.$$

The equations of the slide S_k are

$$(2) \quad \tilde{u} = u, \quad \tilde{v} = v, \quad \tilde{w} = w + k.$$

Since any whirl may be expressed in the form $T_\beta S_k T_\alpha$, the equations of any whirl are

$$(3) \quad \begin{aligned} \tilde{u} &= u + \alpha + \beta, & \tilde{v} &= v \cos (\alpha + \beta) + w \sin (\alpha + \beta) + k \sin \beta, \\ \tilde{w} &= -v \sin (\alpha + \beta) + w \cos (\alpha + \beta) + k \cos \beta. \end{aligned}$$

The only contact transformations of the group of whirl transformations are

$$T_{-\pi/2} S_d T_{\pi/2}, \quad T_{\pi/2} S_d T_{\pi/2}.$$

The first represents a dilatation D_d and the second, which may be written $T_\pi D_d$, represents a dilatation accompanied by a reversal of orientation.

It is then seen that any whirl transformation may be given by

$$S_k D_d T_\alpha;$$

so that the equations of any whirl transformation may be put in the form

$$(4) \quad \begin{aligned} \tilde{u} &= u + \alpha, & \tilde{v} &= v \cos \alpha + w \sin \alpha + d, \\ \tilde{w} &= -v \sin \alpha + w \cos \alpha + k. \end{aligned}$$

It is observed that the group of whirl transformations and the group of motions are isomorphic. These two groups are commutative and together generate a continuous group of six parameters which we call the whirl-motion group G_6 .

The equations of any motion are

$$(5) \quad \begin{aligned} \tilde{u} &= u + \lambda, & \tilde{v} &= v + \mu \cos u + \nu \sin u, \\ \tilde{w} &= w - \mu \sin u + \nu \cos u. \end{aligned}$$

From (4) and (5), we find that the equations of any whirl-motion transformation are

* See Kasner, *The group of turns and slides and the geometry of turbines*, loc. cit.

$$\begin{aligned} \bar{u} &= u + \alpha + \lambda, \\ (6) \quad \bar{v} &= v \cos \alpha + w \sin \alpha + \mu \cos (u + \alpha) + \nu \sin (u + \alpha) + d, \\ \bar{w} &= -v \sin \alpha + w \cos \alpha - \mu \sin (u + \alpha) + \nu \cos (u + \alpha) + k. \end{aligned}$$

This is our fundamental six-parameter group G_6 , containing the whirl group W_3 , and the motion group M_3 as three-parameter subgroups.

2. The invariants for two elements. The angle between two elements is the angle between their lines. The distance between two parallel elements is the distance between their points.

THEOREM 1. Under G_6 we find that (1) two non-parallel elements possess the angle between them as the unique invariant, and that (2) parallel elements are transformed into parallel elements in such a way that the distance between them is preserved.

3. The turbine. A turbine is the series which is obtained by applying a turn T_α to the elements of an oriented circle.

The equations of a turbine are

$$v = a \cos u + b \sin u + r, \quad w = -a \sin u + b \cos u + s.$$

We call (a, b, r, s) a set of turbine coordinates.

Under G_6 , any turbine is converted into a turbine. Thus G_6 induces a correspondence between the turbines of the plane. By (6) we find that this correspondence is given by the equations

$$\begin{aligned} (7) \quad \bar{a} &= a \cos \lambda - b \sin \lambda + \mu \cos \lambda - \nu \sin \lambda, \\ \bar{b} &= a \sin \lambda + b \cos \lambda + \mu \sin \lambda + \nu \cos \lambda, \\ \bar{r} &= r \cos \alpha + s \sin \alpha + d, \\ \bar{s} &= -r \sin \alpha + s \cos \alpha + k. \end{aligned}$$

From the equations of a turbine, we obtain the following theorem:

THEOREM 2. Two non-parallel elements determine a unique turbine.

THEOREM 3. The necessary and sufficient condition that two distinct turbines possess a common element (in which case they have only one common element) is

$$(a_2 - a_1)^2 + (b_2 - b_1)^2 = (r_2 - r_1)^2 + (s_2 - s_1)^2.$$

THEOREM 4. Two turbines have two and only two invariants, namely,

$$(a_2 - a_1)^2 + (b_2 - b_1)^2 \quad \text{and} \quad (r_2 - r_1)^2 + (s_2 - s_1)^2.$$

From this and Theorem 3, it follows that two intersecting turbines

possess the squared distance between their centers as the unique invariant.

THEOREM 5. *A given element and a given turbine possess the unique invariant*

$$[a - (v - r) \cos u + (w - s) \sin u]^2 + [b - (v - r) \sin u - (w - s) \cos u]^2.$$

From this it is seen that a given element E and a given turbine T possess as the unique invariant the squared distance between the element E and that element on T which is parallel to E .

4. Conjugate turbines. Two turbines \bar{T} and T are said to be *conjugate* if they have the same circle as point locus and the elements of the two turbines are symmetrically related to the elements of the circle.

The two turbines $\bar{T}(\bar{a}, \bar{b}, \bar{r}, \bar{s})$ and $T(a, b, r, s)$ are conjugate if and only if

$$\bar{a} = a, \quad \bar{b} = b, \quad \bar{r} = r, \quad \bar{s} = -s.$$

THEOREM 6. *The conjugate turbines of two given turbines do or do not possess a common element according as the two given turbines do or do not possess a common element.*

5. The flat field. The totality of elements determined by the set of all circles which contain a given element is called a *flat field*. The given element is called the *center* (or *central element*) of the flat field.

The equation of a flat field is

$$w = - (v - \bar{v}) \cot \frac{1}{2}(\bar{u} - u) - \bar{w},$$

where $(\bar{u}, \bar{v}, \bar{w})$ are the hessian coordinates of the central element.

Under the whirl-motion group G_6 , any flat field is converted into a flat field. Thus G_6 induces a transformation between the flat fields of the plane. By equation (6), this induced correspondence is given by the equations

$$\begin{aligned} \tilde{u} &= \bar{u} - \alpha + \lambda, \\ (8) \quad \tilde{v} &= \bar{v} \cos \alpha - \bar{w} \sin \alpha + \mu \cos (\bar{u} - \alpha) + \nu \sin (\bar{u} - \alpha) + d, \\ \tilde{w} &= \bar{v} \sin \alpha + \bar{w} \cos \alpha - \mu \sin (\bar{u} - \alpha) + \nu \cos (\bar{u} - \alpha) - k. \end{aligned}$$

It is seen that (8) is a whirl-motion transformation. We call it the *conjugate whirl-motion transformation* \overline{WM} of the whirl-motion transformation WM given by equation (6).

THEOREM 7. *Three elements which are not all on the same turbine and which do not all possess the same direction determine a unique flat field.*

This follows from the equation of a flat field given above.

THEOREM 8. *The turbines which are contained in a flat field are those whose conjugate turbines contain the central element of the flat field.*

THEOREM 9. *Two flat fields whose central elements are not parallel always have a common turbine. Three flat fields, no two of whose central elements are parallel, have a common element or else a common turbine.*

DEFINITIONS. The angle between two given flat fields is the angle between their central elements. Two flat fields are said to be *parallel* or *supplementary (anti-parallel)* according as the angle between them is 0 or π . The *distance* between two parallel flat fields is defined to be the distance between their central elements.

THEOREM 10. *Under G_6 , we find that (1) two non-parallel flat fields possess the angle between them as the unique invariant, and that (2) two parallel flat fields are transformed into parallel flat fields in such a way that the distance between them is preserved.*

THEOREM 11. *A given element and a given flat field possess the unique invariant*

$$\left[(v - \bar{V}) \cos \frac{u - \bar{U}}{2} - (w + \bar{W}) \sin \frac{u - \bar{U}}{2} \right]^2.$$

From this, we find the following geometric interpretation of the invariant: Let E be the given element and F the given flat field. Let the central element of F be denoted by \bar{G} . On the oriented line of E , construct the element G which is in the flat field F . Let l be the line connecting the points of \bar{G} and G . The square of the perpendicular distance from the point of E to the line l is our invariant.

THEOREM 12. *A given turbine and a given flat field possess the unique invariant*

$$[a - (\bar{v} - r) \cos \bar{u} + (\bar{w} + s) \sin \bar{u}]^2 + [b - (\bar{v} - r) \sin \bar{u} - (\bar{w} + s) \cos \bar{u}]^2.$$

From this we find the following geometric interpretation of our invariant. Let T be the given turbine and F the given flat field, let \bar{T} be the conjugate turbine of T , and let \bar{G} be the central element of F . Then our invariant is the square of the distance from the element \bar{G} to the turbine \bar{T} . (See construction in Theorem 5.)