COURANT AND HILBERT ON PARTIAL DIFFERENTIAL EQUATIONS

Methoden der mathematischen Physik. Vol. 2. By R. Courant and D. Hilbert. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 48.) Springer, Berlin, 1937. 16+544 pp.

Thirteen years have elapsed between the publication of the first volume of Courant-Hilbert, *Methoden der mathematischen Physik*, and this concluding second volume. The two volumes are a beautiful, lasting, and impressive monument of what Courant, inspired by the example of his great teacher Hilbert and supported by numerous talented pupils, accomplished in Göttingen, both in research and advanced instruction. Courant came to Göttingen at a time of enormous political and economic difficulties for Germany, on a difficult inheritance, with the day of the heroes, Klein, Hilbert, and Minkowski drawing to a close. But by research and teaching, by personal contacts, and by creating and administering in an exemplary manner the new Mathematical Institute, he did all that was humanly possible to propagate and develop Göttingen's old mathematical tradition. How his fatherland rewarded him is a known story. The publication of the present volume seems to the reviewer a fitting occasion for expressing the recognition his work has earned him in the rest of the mathematical world.

The first volume treated a closed, relatively unramified field: the doctrine of eigen values and eigen functions. It met with enormous success, in particular among the physicists, because shortly after its publication these matters, due to Schrödinger's wave equation, gained an unexpected importance for quantum physics. This second volume covers the theory of partial differential equations in all of its aspects which are of importance for the problems of physics. Its table of contents is therefore necessarily much more variegated, preventing the book from attaining an equally perfect esthetic unity. But it makes up for this by its wealth of material, and it shares the first volume's high didactic accomplishments. Nowadays many mathematical books do not seem to be written by living men who not only know, but doubt and ask and guess, who see details in their true perspective—light surrounded by darkness—who, endowed with a limited memory, in the twilight of questioning, discovery, and resignation, weave a connected pattern, imperfect but growing, and colored by infinite gradations of significance. The books of the type I refer to are rather like slot machines which fire at you for the price you pay a medley of axioms, definitions, lemmas, and theorems, and then remain numb and dead however you shake them. Courant imparts an insight into a situation which has manifold aspects and develops methods without disintegrating them into a discontinuous string of theorems; and nevertheless, the essential results stand out in clear relief. Numerous interesting examples help to enliven and clarify the general theories. In still another respect I found this volume comforting: when one has lost himself in the flower gardens of abstract algebra or topology, as so many of us do nowadays, one becomes aware here once more, perhaps with some surprise, of how mighty and fruitbearing an orchard is classical analysis.

Here follows a brief characterization of what the several chapters of the book contain.

Chapter 1. The usual basic concepts, preliminaries, and isolated elementary methods, namely: Discussion of the manifold of solutions for typical examples, the partial differential equation of a given family of functions, irreducibility of systems to a single equation, geometric interpretation of the equation of first order, complete and singular integrals. Special methods of integration (separation of variables, superposition), the Legendre transformation, linear and quasi-linear equations. The existence theorem for given initial values in the analytic case, treated by means of majorizing power series.

Chapter 2. Theory of characteristics, first illustrated by the quasi-linear equations. Solution of the equation of first order by means of its characteristic strips. The Hamilton-Jacobi theory, including its relationship to the calculus of variations and canonical transformations. The treatment exhibits all sides of these interrelated topics, which were favored subjects for Hilbert's lectures at Göttingen.

Chapters 3–7 deal in the main, though not exclusively, with differential equations of second order.

Chapter 3. The distinction between the hyperbolic, elliptic, and parabolic cases, even for systems of differential equations; the latter receive a more complete treatment than in any other place of which I know. Linear equations with constant coefficients. Plane waves with or without dispersion. Initial value and radiation problems, construction of more general solutions by the method of Fourier integrals, retarded potentials. Illustrative material: heat conduction, wave and telegraph equations. The appendix contains a particularly illuminating exposition of the Heaviside calculus, justifying it to a sufficiently wide extent by Laplace transformations and by a proper application of the process of dislocating paths of integration in the complex plane.

Chapter 4. Potential theory and elliptic equations. Potentials of mass distributions. Green's formula, the Poisson integral, the mean value theorem with its inversion. The boundary value problem is dealt with both by the alternating process and by integral equations. Boundary value problem and parametrix for general elliptic equations. N. Wiener's limit theorems.

The most original part of the book is probably the study of hyperbolic equations contained in Chapters 5–6. The physical ideas of a wave front, of propagation, of rays, the iconal, Huygens' principle and construction have to be rendered in a mathematically tenable and sufficiently general form. Riemann's method of integration, Picard's application of successive approximations, Hans Lewy's integration by introducing the parameters of the characteristic curves as independent variables, and finally Hadamard's profound investigations, which here find a simplified and very lucid exposition, pass muster one after the other and serve to discuss the initial value and the radiation problems, and to settle the questions of unicity and "causality," that is, the question on which part of the initial data (Abhängigkeitsgebiet) the value of the solution at a given point actually depends. Linear equations with constant coefficients, in particular the classical wave equation, receive their due special attention. Huygens' principle is proved both for the initial value and the radiation problem, with emphasis on the difference existing in this respect between even and odd dimensions. H. Lewy's method is used for proving the analyticity of the solutions of analytic elliptic equations. Very interesting are the mean value theorems, in particular one of considerable generality due to Courant's pupil, Asgeirsson, and their applications. The physical side of the problems is not neglected. The relations between physical intuition and mathematical theory are carefully exhibited and illustrated in detail by such important examples as the differential equations of hydrodynamics and of crystal optics.

The seventh chapter deals with the boundary and eigen value problems by means of the calculus of variations. This direct method is here worked out in more detail than has been done before and is applied to transversal deformations and vibrations of plates and to the general 2-dimensional problem of elasticity. The chapter properly concludes with Plateau's problem which has recently entered upon a new phase through J. Douglas' ideas, and to which Courant himself has contributed largely during his last New York years.

The present volume is sprinkled throughout with a wealth of little new illuminating observations which this review had to skip. The author apologizes that lack of time prevented him from fitting out his book with a full sized index of literature and such paraphernalia. The same reason may be responsible for quite a few misprints on which the reader will occasionally stumble. But perhaps even these minor faults deserve praise rather than blame. Although I know that a craftsman's pride should be in having his work as perfect and shipshape as possible, even in the most minute and inessential details, I sometimes wonder whether we do not lavish on the dressing-up of a book too much time that would better go into more important things.

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CONTRIBUTIONS TO THE CALCULUS OF VARIATIONS

Contributions to the Calculus of Variations, 1933–1937. (Theses submitted to the Department of Mathematics at the University of Chicago.) University of Chicago Press, 1937. 7+566 pp.

This volume is the third in the series of contributions to the calculus of variations published by the Department of Mathematics of the University of Chicago since 1930.* It contains thirteen doctoral dissertations, written under the direction of Professors Bliss, Graves, and Reid. The reader acquainted with the work of these mathematicians can make a fair guess as to the way in which responsibility for these papers is to be distributed among them.

The book is uniform in appearance and in style with the earlier ones. About half the papers are followed by a list of references or a bibliography. They are all written with care, conditions are carefully stated, and conclusions are clearly set out. Their contents give a clear picture of the direction in which the work in the calculus of variations has been developed at the University of Chicago in recent years. The contributions contained in the volume are the following:

1. *The problem of Lagrange with finite side conditions*, by J. W. Bower (pp. 1–52).
2. *Fields for multiple integrals in the calculus of variations*, by Byron Cosby, II (pp. 53–84).
3. *The minimum of a definite integral with respect to unilateral variations*, by J. D. Mancill (pp. 85–164).
4. *The Hamilton-Jacobi theory for the problem of Lagrange in parametric form*, by Van Bauman Teach† (pp. 165–206).
5. *Sufficient conditions for a minimum in the problem of Lagrange with isoperimetric conditions*, by I. E. Perlin (pp. 207–242).
6. *A boundary value problem of the calculus of variations*, by E. P. Wiggin (pp. 243–276).

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* The first of these was reviewed in this Bulletin, vol. 38 (1932), p. 617; the second in vol. 39 (1933), p. 641.

† Deceased before the publication of the volume.