ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

1. A. A. Albert: *New proofs of the main theorems on algebras.*

The R. Brauer theorem stating that the direct product of a normal division algebra and its reciprocal algebra is a total matric algebra is used as foundation of the theory. A direct proof of this theorem is obtained as a consequence of the relations between the two regular representations of an algebra. Wedderburn's theorem, stating that if an algebra has a total matric subalgebra this subalgebra is a direct factor, is used then to simplify his determination of the structure of any simple algebra. The Brauer theorem may next be extended immediately to normal simple algebras and applied to show that any scalar extension of a normal simple algebra is normal simple. This treatment results in a considerable simplification of the proofs of the principal theorems on algebras. The repeated application of the Brauer theorem also provides an essential clarification and great condensation of the derivation of the modern theory of normal simple algebras. The exposition will appear in the author's forthcoming Colloquium Lectures. (Received November 21, 1938.)

2. C. B. Allendoerfer: *An n-dimensional Gauss-Bonnet theorem.* Preliminary report.

The classical Gauss-Bonnet theorem is generalized to conformally flat spaces S_n of any even dimension. Let V_{n-1} be a closed subspace of S_n which can be mapped into a unit sphere by a conformal mapping which carries S_n into a flat space. Then an integral (whose integrand is essentially the mean normal curvature of V_{n-1}) over V_{n-1} plus an integral (depending solely on the metric of S_n) throughout the interior of V_{n-1} is equal to the area of an ordinary unit sphere of $n-1$ dimensions. For S_n of odd dimension, the formula omits the integral over the interior of V_{n-1} and modifies the integral over V_{n-1}. (Received November 23, 1938.)

3. C. J. Blackall: *Volume integral invariants of non-holonomic systems.*

In this paper necessary and sufficient conditions are given that a non-holonomic dynamical system, in which the applied forces are functions of the position coordinates only, have a volume integral invariant. Special attention is paid to the case in which the integrand of the invariant integral is a function of the position coordinates only. An example is given to prove that not all analytic conservative non-holonomic systems admit volume integral invariants in which the integrand is non-trivial and analytic. (Received November 21, 1938.)

4. L. M. Blumenthal and G. R. Thurman: *Spheroidal and pseudo-spheroidal spaces.*

Denote by S a semimetric space of finite diameter d. If r is a positive number, n a positive integer, $F(pq/r)$ a real monotonic decreasing function defined over the distance set of S, with $F(0)=1$, $F(d/r)=-1$, then the system $(S, F; n, r)$ is a spheroidal space provided a certain set of five metric conditions is satisfied. A semimetric space is pseudo-$(S, F; n, r)$ provided each set of $n+2$ of its points is congruently embeddable in $(S, F; n, r)$ while the whole space is not. This paper develops the geometry of spheroidal and pseudo-spheroidal spaces. The principal theorem asserts that if P is any pseudo-$(S, F; n, r)$ space containing more than $n+3$ points, and $p, q \in P$ implies $pq \neq d$, then for every integer k and every set of $k+1$ points of P, the determinant $|F(p_ip_j/r)|$, $(i, j=1, 2, \cdots, k+1)$, has (upon multiplication of appropriate rows and the same numbered columns by -1) all elements outside the principal diagonal equal to $-1/(n+1)$. Thus P is essentially equilateral. Since the n-dimensional spherical surface of radius r is a spheroidal space $(F(pq/r)\equiv\cos pq/r)$, this result contains, as a special case, the solution of the problem proposed by Menger of characterizing pseudo-spherical spaces. (Received November 7, 1938.)

5. R. P. Boas (National Research Fellow): *The closure of translations of almost periodic functions.*

Let $f(x)\sim\sum_{n=-\infty}^{\infty}a_ne^{i\lambda_nx}$ be a uniformly almost periodic function in which no a_n is zero. Let $g(x)\sim\sum_{n=-\infty}^{\infty}b_ne^{i\mu_nx}$ be another uniformly almost periodic function. Then it is possible to approximate to $g(x)$, uniformly on $(-\infty, \infty)$, by finite sums of the form $\sum c_kf(x+u_k)$, if and only if every μ_n occurs in the sequence $\{\lambda_n\}$. Similar theorems can be established for other classes of almost periodic functions. (Received November 15, 1938.)

6. R. P. Boas (National Research Fellow) and Salomon Bochner: *On a theorem of M. Riesz for Fourier series.*

A theorem of M. Riesz states that if $f(x)$ is a function of L^p, $(p>1)$, of period 2π, and $S^{(n)}(x)$ is the nth partial sum of its Fourier series, then $\int_0^{2\pi}|S^{(n)}(t)|^pdt\leqq A_p\int_0^{2\pi}|f(t)|^pdt$, where A_p depends only on p. This theorem is generalized to the case of abstract-valued functions with values in a space l^r, $(r>1)$. Written explicitly, the result is that if $\{f_\nu(t)\}$ is a sequence of integrable functions, of period 2π, and $S_\nu{}^n(t)$ is the nth partial sum of the Fourier series of $f_\nu(t)$, then $\int_0^{2\pi}\{\sum_{\nu=1}^{\infty}|S_\nu{}^{(n)}(t)|^r\}^{p/r}dt\leqq A_{p,r}\int_0^{2\pi}\{\sum_{\nu=1}^{\infty}|f_\nu(t)|^r\}^{p/r}dt$ where $A_{p,r}$ depends only on p and r. The case $r=2$ has been discussed by Paley and Littlewood and by Zygmund (see Proceedings of the Cambridge Philosophical Society, vol. 34 (1938), p. 128). The result is false when $r=1$, as has been shown recently by S. Bochner and A. E. Taylor. (Received November 15, 1938.)

7. J. W. Calkin: *An abstract self-adjoint boundary value problem.*

Let H be a closed symmetric transformation in Hilbert space $\mathfrak{H}$, and let A be a reduction operator for H^* (see the author's paper, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 38–42, Definition 1). Let $Af \in \mathfrak{N}$ be a boundary condition defining a self-adjoint extension $H(\mathfrak{N})$ of H (loc. cit., Definition 2), and let λ belong to the resolvent set of $H(\mathfrak{N})$. Assuming a solution f of the equation $H^*f-\lambda f=f^*$, where f^* is an arbitrary element of $\mathfrak{H}$, and a complete orthonormal set $\{u_n\}$ in the manifold of zeros of $H^*-\lambda I$, the author determines an expansion

$g=f+\sum_n a_n u_n$ for the unique element g of $\mathfrak{D}(H^*)$ such that $H^*g-\lambda g=f^*$, $Ag \in \mathfrak{N}$. The result provides an algorithm by means of which an expansion for the Green's function associated with each of a wide class of self-adjoint boundary value problems can be determined. (Received November 22, 1938.)

8. R. H. Cameron and Norbert Wiener: *Convergence properties of analytic functions of Fourier-Stieltjes transforms.*

Let $F(x)$ be the Fourier-Stieltjes transform of the function of bounded variation $f(u)$; let $h(u)$, $g(u)$, $s(u)$ be the discrete, absolutely continuous, and singular parts of $h(u)$; and let $H(x)$, $G(x)$, $S(x)$ be the three corresponding parts of $F(x)$. The authors define two norms $T_\theta^*[F]$ and $T_\theta^{**}[F]$ for $F(x)$ and for $0<\theta\leqq 1$. The first of these norms is twice the sum for all integers n of the θ powers of the total variation of $f(u)$ on the interval $n\leqq u<n+1$. The second norm is the above expression for $g(u)+s(u)$ plus twice the sum of the θ powers of the jumps of $h(u)$. This paper shows that if $A(z)$ is a single-valued analytic function in a certain region R and $F(x)$ is of finite θ-norm, then $A[F(x)]$ is also of finite θ-norm, the two θ-norms being of the same type. The region R contains all points within $\{T_\theta[S]\}^{1/\theta}+\delta$ of values of $F(x)$. The same theorem holds for n-valued functions $A(z)$ if $G(x)$ is identically zero. If $G(x)$ is not identically zero, an extra condition is necessary to insure that $A(G(x_1))$ and $A(G(x_2))$ are on the same branch of $A(z)$ when x_1 and x_2 are corresponding points near $+\infty$ and $-\infty$. (Received November 26, 1938.)

9. H. H. Campaigne: *Hypergroup structures.*

The purpose of this paper is to study the conditions under which the subhypergroups of a hypergroup constitute a structure. If G is a finite group with a conjugation among its elements, then a subgroup F is said to be appropriate if it is closed under the conjugation. The set of appropriate subgroups of G forms a structure $\mathcal{G}$. The classes $\{b\}$ of conjugate elements of G form a hypergroup $\{G\}$, and if F is an appropriate subgroup, then and only then is $\{F\}$ a subhypergroup of $\{G\}$ containing $\{e\}$ (e is the identity of G). The set of such subhypergroups of $\{G\}$ is a structure, and is abstractly identical with $\mathcal{G}$. If the conjugation is defined in terms of a subgroup A, that is, if $a\sim b$ if and only if ab^{-1} is in A, then the appropriate subgroups are those containing A. If the conjugation in G is one determined by a group Γ of the automorphisms of G, the appropriate subgroups are those invariant under the automorphisms of Γ. If Γ is the group of inner automorphisms of G, then G is simple if and only if $\{G\}$ has no proper subhypergroups. (Received November 22, 1938.)

10. L. W. Cohen: *On imbedding a space in a complete space.*

Let S be a set of elements p, and assume that for each p in S a sequence of sets $U_n(p)$ is defined which satisfies the following conditions: (1) $\prod_{n=1}^{\infty} U_n(p)=p$; (2) $U_n(p)U_m(p)\supset U_k(p)$ for each p and some $k=k(p, n, m)$; (3) for each p in S and each n there are integers $\lambda(n)$ and $\delta(p, n)$ such that if $U_{\delta(p,n)}(q)U_{\lambda(n)}(p)\neq 0$, then $U_{\delta(p,n)}(q)\subset U_n(p)$. Under these conditions, there is a complete regular Hausdorf space S^* satisfying the first denumerability axiom and a mapping f on S to a subset $f(S)\subset S^*$, such that f and f^{-1} are single-valued and continuous and $f(p_k)$ is a convergent sequence in S^* if p_k is a Cauchy sequence in S. The sequence p_k is called a Cauchy sequence in S if for each n there are q_n in S and $k_n>0$ such that $p_k\subset U_n(q_n)$ for $k\geqq k_n$. A complete space is one in which every Cauchy sequence has a limit. The question of the metrizability of S is open. (Received November 16, 1938.)

11. Richard Courant: *Conformal mapping of multiply connected domains.*

The solution of Plateau's problem leads, as was first emphasized by Douglas, to mapping theorems if the prescribed contour lies in a plane. By the methods for the solution of Plateau's problem developed by the author, new mapping theorems can be obtained; for example, each k-fold connected domain can be mapped conformally on a domain consisting of k concentric circles in different sheets connected by $2k-2$ branch points, so that $k+1$ branch points are prescribed. It is also possible to choose all the circles as unit circles if the branch points are left free; then one may prescribe the image of one point on the boundary of each circle. (Received November 21, 1938.)

12. L. A. Dye: *Involutorial transformations determined by two systems of lines.*

Two transformations of general order are generated by associating the surfaces of a pencil with the lines of two systems. Through a general point P of space passes one surface of the pencil and one transversal of the associated pair of lines. The residual intersection of this transversal with the surface is the image of P in an involutorial transformation. The transformations, which are regular, have fundamental curves of both the first and second species. (Received November 21, 1938.)

13. W. W. Flexner: *The character group of a relative homology group.*

Let G and H be abelian groups, one discrete and the other locally bicompact, which are character groups of one another with respect to the group of real numbers reduced modulo 1. If Γ is a subgroup of G (written $\Gamma \subset G$), denote by (H, Γ) the group of all elements of H mapped on the identity by all elements of Γ. Then if $\Gamma \subset \mathfrak{G} \subset G$ and $\theta \subset \mathfrak{H} \subset H$, each subgroup being closed in the group containing it, and if $\mathfrak{G} = (G, \theta)$ and $\mathfrak{H} = (H, \Gamma)$, it is proved that $\mathfrak{G}/\Gamma$ and $\mathfrak{H}/\theta$ are character groups of one another. If G^s is the direct product $G \times \cdots \times G$ to s factors, s is finite, $F \subset H^s$, and t is a positive integer less than s, then it is shown that $(G^s, F)^*$ is isomorphic to (G^t, F^*), where the star means projection on G^t or H^t. Two more lemmas, similar to this, are proved. If L is a subcomplex of a finite complex K, it then follows that the group of r-G-cycles of L independent with respect to bounding on L but bounding in K is the character group of the group of r-H-cocycles of L modulo the subgroup of those of them which are r-H-cocycles of K. (Received November 22, 1938.)

14. L. R. Ford: *Normal families and differential equations.*

This is a discussion of the role which the theory of normal families might be made to play in proving the existence of solutions of differential equations in the analytic cases. The equation $w''+Q(z)w=0$, where $Q(z)$ is analytic, is studied explicitly. (Received November 22, 1938.)

15. K. O. Friedrichs and J. J. Stoker: *The nonlinear boundary value problem of the buckled plate.*

A constant pressure is applied along the edge and in the plane of a thin circular plate. The mathematical formulation of the problem leads to two nonlinear differential equations for deflection and stress functions with appropriate boundary conditions. Beyond a certain critical value of the pressure the problem has three different solutions: the state with no deflection and two buckled states. In order to obtain the latter, a method similar to the perturbation procedure is employed. The results show that the plate beyond the critical pressure is very much stiffer than its one-dimensional analogon, the beam under axial thrust. (Received November 26, 1938.)

16. Saul Gorn: *Some theorems on dimensions in partially ordered sets.*

These results enlarge upon those mentioned in this Bulletin, abstract 44-5-201. A dimension is a real function on a partially ordered set with $d(a)+d(b)=d(ab)+d(a+b)$ when ab and $a+b$ exist. It is strong if $a>b$ implies $d(a)>d(b)$; weak if $a>b$ implies $d(a)\geqq d(b)$; normalized if $d(0)=0$, $d(1)=1$. The results are: (1) that a reducible complemented modular lattice cannot have a unique normalized dimension; (2) the obvious corollary for Boolean algebras; (3) that if L' is a projective extension of the incidence geometry L with the same zero, the ratio of the rank of an element in L to its rank in L' is constant; (4) an obvious generalization of (3), if L and L' do not have the same zero; (5) that if $r(x)$ is a normalized weak dimension in a Boolean algebra, the elements of dimension one (zero) form a π-ideal (σ-ideal), called the determined ideal $\mathfrak{a}$; (6) that there is a 1–1 correspondence between the possible $r(x)$ determining $\mathfrak{a}$ and the strong dimensions in the quotient Boolean algebra relative to $\mathfrak{a}$; (7) that $r(x)$ is unique for $\mathfrak{a}$ if and only if $\mathfrak{a}$ is prime. (Received November 7, 1938.)

17. Saul Gorn: *The complete envelope of a non-normal partially ordered set.*

A generalization of the result mentioned in this Bulletin, abstract 44-5-201, is obtained by *ideal extension*. A partially ordered set is non-normal if the sufficient conditions for existence of sum and product are not valid. An extension L' of L is called conservative if $a<b$ in L' implies $a<b$ in L for any elements in L. Two operations on subsets of L are introduced which generate and permit the definition of complete ideals. When iterated alternately, they have the formal properties of the star operator for linear transformations in Hilbert space. The extension L' is a conservative extension of L if and only if the complete ideals of L are obtained from those of L' by intersection with L. The complete ideals in a general (non-normal) partially ordered set form the minimal conservative extension of L. The condition that the extensions be conservative is necessary even if L is normal, as is shown by an example. (Received November 7, 1938.)

18. D. W. Hall (National Research Fellow): *Some applications of true secondary elements.*

Let C be a cyclicly connected continuous curve. Then, as has been previously shown by the author (see this Bulletin, abstract 44-5-205), certain closed subsets of C may be distinguished and known as true secondary elements. Let C have the property that for every true secondary element E of C the set $C-E$ has but a finite number of components. Let P be any one of the following properties: (1) that of being a regular curve, (2) that of being a rational curve, (3) that of being hereditarily locally connected. Among the results of the present paper is the following theorem: A necessary and sufficient condition that C have property P is that each component of every true secondary element E of C also have property P. (Received November 21, 1938.)

19. Israel Halperin: *The spectrum of a continuous flow.*

J. von Neumann has shown that for a differentiable ergodic flow the point spectrum is a modulus and the continuous spectrum is either absent or fills the entire real axis (see Annals of Mathematics, (2), vol. 33 (1932), p. 635). He conjectures that the same results hold for all ergodic flows. The writer establishes this conjecture for continuous ergodic flows. (Received November 23, 1938.)

20. O. G. Harrold: *On the non-existence of certain continuous transformations defined on an arc.*

It is well known that there exist continuous transformations defined on an arc such that the inverse of each point in the image space consists of exactly k points, for $k=1, 3, 4, 5, \cdots$. In this note it is shown that for $k=2$ no such transformation exists. A fundamental lemma needed for the proof is this: It is impossible to map continuously a Cantor discontinuum on a space A containing an arc such that every point of A has exactly two inverse points. (Received November 22, 1938.)

21. Albert Heins: *Some remarks on the solution of partial difference equations.*

Various methods of solving partial difference equations under given conditions are studied. Methods differing from the classical methods are considered. (Received November 21, 1938.)

22. M. R. Hestenes: *Generalized problem of Bolza in the calculus of variations.*

In the present paper the author studies the problem of minimizing a function $I=g(a)=\int_{x_1}^{x_2} f(a, x, y, y')dx$ in a class of arcs a_h, $y_i(x)$, $(x_1 \leqq x \leqq x_2;\ h=1, \cdots, r;\ i=1, \cdots, n)$, in axy-space satisfying the conditions $\phi_\gamma(a, x, y, y')=0$, $x_s=x_s(a)$, $y_i(x_s)=y_{is}(a)$, $(s=1, 2)$, and $I_\rho=0$, where the functions I_ρ are of the same form as I. This problem contains as immediate special cases the problems of Bolza and Mayer in the Bliss and Morse forms, the isoperimetric problem, the problem of minimizing a function of integrals, and other problems. Sufficient conditions for strong relative minima are established. These conditions are applicable at once to the above special cases. The method used is an extension and a simplification of one previously used by the author (Transactions of this Society, vol. 42 (1937), pp. 141–153) in sufficiency proofs for the problem of Bolza. (Received November 21, 1938.)

23. M. R. Hestenes and W. T. Reid: *A note on the Weierstrass condition in the calculus of variations.*

The purpose of the present note is to prove the following theorem and its analogue in the problem of Bolza. Suppose that the determinant $|f_{p_i p_k}|$ associated with an integrand function $f(x, y, p)$ is different from zero on a region N of admissible elements $(x, y, p)=(x, y_1, \cdots, y_n, p_1, \cdots, p_n)$. If the Weierstrass E-function determined by $f(x, y, p)$ satisfies the condition $E(x, y, p, q) \geqq 0$ for all (x, y, p) on N and all (q) such that (x, y, q) is admissible, then $E=0$ only if $(q)=(p)$. The corresponding theorem for the problem of Bolza in parametric form is also given. (Received November 21, 1938.)

24. T. R. Hollcroft: *Cuspidal limits for plane curves.*

Limits to the number of cusps of a plane curve system of order n and genus p with distinct nodes and cusps are implied by Plücker's equations. These limits were expressed in explicit form by Lefschetz in 1913. In the same paper, Lefschetz established the existence of maximal cuspidal curve systems satisfying Plücker limits for any n and $p \leqq p_0$ (p_0 is the lowest genus of the curve of least class for a given order). These curve systems are all regular. There is not only no similar proof for $p > p_0$, but Zariski has proved that certain systems satisfying the Plücker limits for $p > p_0$ do not exist. In this paper it is shown, for n sufficiently large, that regular maximal cuspidal curve systems exist for genera up to $p_r > p_0$. It is also true that regular curve systems with only cusps exist for $p_r' < (n-1)(n-2)/2$. Within the interval $p_r' - p_r$ lie the genera of all irregular systems, although not all of these systems are irregular. Finally, within this interval $p_r' - p_r$, for n sufficiently large, there are values of p for which cuspidal limits holding for all values of n cannot be found. (Received November 21, 1938.)

25. Charles Hopkins: *Rings with minimal condition for left ideals.*

Let A denote a ring with minimal condition for left ideals (*MLI* ring). It is proved that the radical R of A is nilpotent; the sum of all minimal left ideals of A is a two-sided ideal M and the direct sum of a finite number of left ideals of A; the sum of all minimal nilpotent left ideals is a two-sided ideal of A. Every *MLI* ring is shown to be the sum of a nilpotent and an idempotent *MLI* ring. The structure of these two special types of *MLI* rings is investigated. In particular, it is shown that for a nilpotent *MLI* ring B the minimal (maximal) condition for left ideals implies the minimal (maximal) condition for subrings of B. For a non-nilpotent *MLI* ring J the existence of either a right-hand or a left-hand identity implies the existence of a composition series of left ideals of J. (Received November 21, 1938.)

26. Dunham Jackson: *Linear dependence of polynomials on algebraic curves in space.*

It has been shown previously that the number of polynomials of the nth degree in a system of polynomials orthogonal on the curve of intersection of two quadrics cannot exceed 4. It is pointed out now that for any algebraic curve in space the corresponding number is bounded as n becomes infinite, and explicit upper bounds for it are obtained in a variety of cases. (Received November 14, 1938.)

27. Fritz John: *An identity for the integrals of a function over two concentric ellipsoids.*

Let E_1 and E_2 be two ellipsoids in 3-space with center at the origin and with tangential equations $\sum_{i,k} a_{ik}u_iu_k = 1$ and $\sum_{i,k} b_{ik}u_iu_k = 1$, respectively. In this paper there is derived an explicit expression for $\int_{E_1} f d\omega - \int_{E_2} f d\omega$ with an arbitrary function f in terms of the values of $\sum_{i,k} (a_{ik} - b_{ik})\partial^2 f/\partial x_i \partial x_k$ in the convex hull of E_1 and E_2. Here $d\omega$ denotes the affine element of surface of the ellipsoids. This includes, in the special case $a_{ik} - b_{ik} = \delta_{ik}$, $\Delta f = 0$, the result that the mean values of a potential function over confocal ellipsoids are the same. (Cf. Asgeirsson, Mathematische Annalen, vol. 113; Nicolesco, Mathematiche Annalen, vol. 114.) (Received November 22, 1938.)

28. B. W. Jones: *A note on a minimum for positive quadratic forms.*

While the Eisenstein reduction of positive n-ary quadratic forms $f = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ is concerned with minimizing the a_{ii} of the form, the Selling reduction minimizes the sum $s = \sum_{i \leqq j}^{n} a_{ij}$. This amounts to what might be called a minimum mean of the coefficients. By means of an extension of Hermite's reduction one finds for the minimum s, in the case of integral forms, a simple upper bound depending only on the determinant of the form and n. There is no such upper bound for non-integral forms. (Received November 21, 1938.)

29. J. L. Kelley: *Concerning a decomposition of compact continua.*

For a given compact continuum M define an F-set in M to be an end point, a cut point, or a nondegenerate Mp, where p is a non-cut point and Mp is the set of all points not separated from p in M by any point. This differs from R. L. Moore's definition of simple link only in the classification of a certain set of cut points. The present definition is chosen so that if M is locally connected, the F-sets are simply the cyclic elements. It is shown that M is the sum of its F-sets, that a point not belonging to a nondegenerate F-set is a regular point in the sense of Menger-Urysohn, and that many other cyclic element theorems hold. Also the following fixed set theorems are obtained: If T is a continuous transformation, $T(M) \subset M$, (1) there exists a continuum $N \subset M$ such that $T(N) \supset N$ and N is not separated in M by any point of M, (2) there exists a fixed point or an F-set F such that $F \cdot T(F)$ contains a nondegenerate continuum, (3) there exists a compact set R such that $T(R) \equiv R$ and R is a subset of an F-set. (Received November 21, 1938.)

30. A. J. Kempner: *On the roots of algebraic equations with complex coefficients.*

Theorem 1. Given $f(z) = f(x+iy) = \sum a_k z^k$ with complex coefficients, and a line $y = mx + b$, m, b real, by rational operations on the a_k and their conjugate complex $\bar{a}_k$ and on m and b, it is possible to factor $f(z)$ into $f_1(z) \cdot f_2(z)$, such that the roots of $f_1(z) = 0$ are exactly those roots of $f(z) = 0$ lying on $y = mx + b$ or symmetric in pairs to $y = mx + b$. For this and the following theorems, simple algebraic proofs are given. For a special case, a function-theoretic proof was given by Milne and Thompson, Mathematical Gazette, vol. 21 (1937), pp. 288–289. Theorem 2. Necessary and sufficient conditions that (a) some roots of $f(z) = 0$ lie on the axis of reals, or (b) that some couples of roots be conjugate complex, or that (a) and (b) both hold, is the vanishing of the determinant of order $2n$ representing the resultant of $\phi_1(z) = \sum \alpha_k z^k$, $\phi_2(z) = \sum \beta_k z^k$, $(k = 0, \cdots, n, \alpha_k + i\beta_k = a_k)$. Theorem 1 may be extended from a straight line to a real polynomial curve $y = \psi(x) = \sum c_\mu x^\mu$ in the z-plane, as well as to other types of curves. If symmetry of a pair of points in the z-plane with respect to $y = \psi(x)$ is interpreted in the sense of Schwarz' Spiegelungsprinzip, the following theorem may be stated. Theorem 3. By rational operations on the a_k, $\bar{a}_k$, and c_k it is possible to factor $f(z) = f_1(z) \cdot f_2(z)$ so that the roots of $f_1(z) = 0$ are the roots of $f(z) = 0$ lying on $y = \psi(x)$ or in pairs symmetric to $y = \psi(x)$. Other theorems and applications are derived. (Received November 22, 1938.)

31. Alfred Korzybski: *Extensionalization in mathematics, mathematical physics, and general education. General semantics.* II.

The following topics are treated in this paper: intuition, reason, and faith in mathematics and life; fundamental neurological factors involved (never suspected before) which account for the cultural and sanity empirical bearings of mathematical methods; geometry, analysis, and analytical geometry, and their neurological values; structure of languages in general and mathematics in particular; new nonelementalistic definition of number; the Einstein theory; predicatability, based on proper evaluation, ultimately depends on similarity of structure between map-territory, language-facts, achieved in mathematics; necessary revision of the structure of language after mathematical patterns; structural revision of languages introduces needed thelamic factors into symbolism; the automatic empirical results achieved in general semantics, psychiatry, and education, and their lasting effects; the human and cultural value of the application of elementary mathematical methods toward general sanity and its automatic effect on more efficient education, and so on. (Received November 22, 1938.)

32. O. E. Lancaster: *Orthogonal polynomials defined by difference equations.*

This paper is devoted to a study of polynomial solutions of difference equations of the form $(ax^2+bx+c)\Delta_h^2 y(x)+(dx+f)\Delta_h y(x)+g\lambda y(x+h)=0$, where λ is a parameter, h is the interval of difference, and a, b, c, d, f, and g are constants. After a new definition of an adjoint equation is made, it is proved that every second order difference equation can be put in a self-adjoint form. Then it is shown that solutions corresponding to characteristic values of λ are orthogonal (in a sense defined) on some interval. Special properties of these orthogonal polynomials are developed. Included among them is a difference form and a recurrent relation for the polynomials. The results are shown to reduce, in the limit as $h\rightarrow 0$, to the known facts for differential equations. The general theory is illustrated by polynomials analogous to the Legendre and Hermite polynomials, which were studied by Jordan and Greenleaf (Annals of Mathematical Statistics, vol. 3 (1932), pp. 204–357), and also by polynomials which are developed in this paper. (Received November 18, 1938.)

33. E. H. Larguier: *A matrix theory of n-dimensional measurement.*

The author extends to n dimensions some of the results obtained by A. H. Copeland (*A matrix theory of measurement*, Mathematische Zeitschrift, vol. 37 (1933), pp. 542–555) who has given a precise statement of the fundamental assumptions of a matrix theory of measurement which would serve for a one-dimensional theory of probability. Somewhat later, Copeland (*The probability limit theorem*, Duke Mathematical Journal, vol. 2 (1936), pp. 171–176) showed that the assumption that physical measurements behave in accordance with the probability limit theorem does not involve any inconsistency in the mathematical sense. This same notion has been extended by the author to include the probability theorem for n dimensions. Incidentally, a simplification of one of Copeland's theorems (p. 174) is given in the present paper. (Received November 15, 1938.)

34. R. G. Lubben: *Composition points in abstract spaces.*

A space T is defined for the aggregate of all composition points of a space S which is a space H Fréchet; as a consequence of this definition, T is a space $\mathfrak{H}$. (1) In order that $M\subset T$ be perfectly compact in itself and that no two of its points intersect, it is necessary and sufficient that M determine an amalgamation point and that it be upper semicontinuous relative to S; the aggregate of atomic elements of T satisfies this condition. (2) In order that it be possible to regard each element of T which is "regular relative to S" as the amalgamation of its "regular-atomic" pieces, it is necessary and sufficient that (a) each element of T intersect a regular atomic element of T and (b) two regular (relative to S) elements of T intersect only if they intersect in a regular (relative to S) element of T. (3) If the condition in (2) is satisfied, the aggregate of regular-atomic composition points satisfies that of (1); and the regular composition points of S admit a theory similar to that in abstracts 43-9-344 and 44-1-32. This is not true for all the composition points; in general the atomic points are finer than the regular-atomic points. (Received November 21, 1938.)

35. W. H. McEwen: *Note concerning a generalization of Bernstein's theorem.*

Zygmund has generalized S. Bernstein's theorem for trigonometric polynomials $T_N(\theta)$, $(\theta=2\pi x)$, as follows: $\|T_N'\|_p \leqq N\|T_N\|_p$ where $\|f\|_p=[\int_0^1|f|^p dx]^{1/p}$, $(p\geqq 1)$. Hille, Szegö, and Tamarkin have made a corresponding generalization of Markoff's theorem in the case of polynomials $P_N(x)$, $\|P_N'\|_p \leqq CN^2\|P_N\|_p$. In the present paper the author observes that similar generalizations can be made also in the case of sums $S_N(x)$ of Birkhoff type, associated with a given regular linear differential system of the nth order, $L(u)+\lambda u=0$, $W_j(u)=0$, $(j=1, 2, \cdots, n)$. In general, $\|S_N'\|_p \leqq C'N^2\|P_N\|_p$, but suitable additional restrictions on the boundary conditions give the relation $\|S_N'\|_p \leqq C''N\|S_N\|_p$. (Received November 21, 1938.)

36. E. J. McShane: *Curve-space topologies associated with variational problems.*

The usual definition of a weak relative minimum for parametric problems in the calculus of variations involves inequalities of the form (a) $|y(t)-\gamma(t)|<\epsilon$, (b) $|y'(t)-\eta'(t)|<\epsilon$, where $y=\gamma(t)$ (that is, $y^i=\gamma^i(t)$), $(0\leqq t\leqq b)$, is some favored representation (for example, in terms of arc length) of a curve Γ. It would be esthetically preferable to base the definition of weak relative minimum on a concept of neighborhood in curve-space. In this note six topologies are defined, and their interrelations studied. Two of these lead to a notion of weak relative minimum equivalent to the usual one; a third (defined by replacing (b) by the condition that the angle between $y'(t)$ and $\gamma'(t)$ be not greater than $\epsilon_1<\epsilon$ for almost all t) is equivalent if Γ is of class D', but not if it is merely rectifiable. The other three topologies are mutually equivalent but weaker than the first. They are the weakest topologies in which every integral $\int F(y, y')dt$ is a continuous function of curves. (Received November 21, 1938.)

37. E. J. McShane: *On the uniqueness of the solutions of differential equations.*

If $y_1(x)$ (that is, $(y_1^1(x), \cdots, y_1^n(x))$ and $y_2(x)$ are solutions of the equations $y^i=f^i(x, y)$, $(a\leqq x\leqq b)$, and for some x_0 in $[a, b]$ these solutions coincide, they still may not be identical. However, if there is a summable function $M(x)$ such that (1) $\eta^i\{f^i(x, y+\eta)-f^i(x, y)\}\leqq M(x)|\eta|^2$ (where $|\eta|^2=\eta^i\eta^i$) for all x in $[a, b]$ and all y and all η, then the solutions will be identical for $x_0\leqq x\leqq b$. If in (1) the sign $\leqq$ is replaced by $\geqq$, the solutions will coincide for $a\leqq x\leqq x_0$. Two special cases are obtained, the standard theorem on uniqueness in which it is assumed that $|f(x, y+\eta)-f(x, y)|\leqq M(x)|\eta|$, and a theorem of Graves in which $n=1$ and $f(x, y)$ is monotonic decreasing as a function of y. (Received November 21, 1938.)

38. E. J. McShane: *The Jacobi condition and the index theorem in the calculus of variations.*

Bliss's treatment of the Jacobi condition (1916) made use of a class of solutions $\eta^i(t)$ of the Jacobi equations which were called "normal," satisfying $\eta^i\dot{y}^i=0$. Later authors replaced this condition by others, in some cases to make the treatment of the parametric problem resemble more closely the non-parametric theory, and in another (M. Morse) to facilitate the proof of the index theorem. The purpose of the present paper is to show that a modification of Bliss's treatment can be devised which (a) is invariant under change of parameter and of coordinates, (b) yields the Jacobi necessary condition and the tests for conjugacy in the non-parametric problem as ready consequences of those for the parametric problem, and (c) is not less suited than Morse's for the proof of the index theorem. The "normal" solutions of Bliss are replaced by "p-normal," solutions, satisfying $\eta^i(t)p_i(t)=0$, where p_i is covariant under change of coordinates, is at most multiplied by a factor $k(t)\neq 0$ under change of parameter, and satisfies $p_i\dot{y}^i\neq 0$. (Received November 21, 1938.)

39. Saunders MacLane: *Modular fields.* I. *Separating transcendence bases.*

If $K\supset L$ are fields of characteristic p, a separating transcendence basis T of K over L is a set of elements of K algebraically independent over L, such that every element of K is separable and algebraic over $L(T)$. If K is a function field of n variables over a perfect coefficient field L, then K always has a separating transcendence basis over L. The same result is demonstrated if the base field L is itself a function field of one variable over a perfect coefficient field. For more general fields L, necessary and sufficient conditions for the existence of a separating transcendence basis for K/L are given. They involve the consideration of a class of extensions K/L which preserve p-independence, in the sense that elements of L which are p-independent in L must remain p-independent in K. In consequence of these considerations, it is shown that if K has a finite separating transcendence basis over L and if M is a field between K and L, then M also has a separating transcendence basis over L. Finally an example is given to show that a field K need not have a separating transcendence basis over its maximal perfect subfield P except when the transcendence degree of K over P is 1. (Received November 22, 1938.)

40. Saunders MacLane and O. F. G. Schilling: *Infinite number fields with Noether ideal theories.*

This paper is concerned with the explicit construction of all fields and rings with certain given arithmetic properties. For instance, a field K is divisor-discrete if every valuation of K is discrete, and is divisor-finite if every element of K has a value different from zero in only a finite number of non-equivalent valuations. All fields K which are both divisor-discrete and divisor-finite are found; they include finite algebraic number fields, certain infinite such fields, and certain function fields with coefficient fields which are absolutely algebraic. A domain of integrity has a Noether ideal theory if every ideal has a unique decomposition into a product of prime ideals and if divisibility of ideals implies factorizability. A sample theorem is the following: A field K is divisor-discrete and divisor-finite if and only if every integrally closed subring J of K has a Noether ideal theory. (Received November 14, 1938.)

41. H. M. MacNeille: *A constructive theory of integration.*

This paper presents a construction, directly from step functions, of an integral analogous to Bochner's generalization of the Lebesgue integral. Let S be the class of finite-valued step functions from a Boolean ring K of point sets (the properties of K are described in the writer's paper, Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 188–193) to a linear, metric, but not necessarily complete, space J; then S is a linear space. An integral is defined and its properties are developed in the space S of step functions. A metric is defined in S. The metrics of the spaces L^p are among the possible metrics for S. In general, the set S will not be complete under the metric defined for it. The set S and the definition and properties of its integral are extended to a complete metric space by means of the absolutely convergent sequences in S. The use of absolute convergence simplifies the theory by making the selection of subsequences unnecessary. This method of construction leads directly to such topics as derivatives, variation, spaces L^p, absolute continuity, and Fubini's theorem. Neither a partial order in J nor a topology in the set of points from which the elements of K are drawn is required by this method. (Received November 22, 1938.)

42. W. A. Manning: *A note on transitive groups with regular subgroups of the same degree.*

A simply transitive group of degree n (n not a prime) in which there is a regular abelian subgroup H of order n, one of whose Sylow subgroups is cyclic, is imprimitive. The proof of this theorem by H. Wielandt is of admirable brevity, but depends upon several propositions laid down by I. Schur in his memoir on that less general case in which H is cyclic. The whole matter can be greatly simplified by introducing the concept of double cosets. For example, a primary complex of H is the crosscut of H and $G_a s G_a$, where G_a is a subgroup that fixes one letter a of G, and s is any permutation of G. The proof of Wielandt's theorem is obtained very simply without the use of any representation theory and without the use of any ring, heretofore considered indispensable. (Received November 19, 1938.)

43. M. H. Martin: *Restricted problems in three bodies.* Preliminary report.

A restricted problem in n bodies is obtained when the motion of k, $(1 \leqq k \leqq n-1)$, of the bodies is specified beforehand and the motion of the remaining $n-k$ bodies is required under the hypothesis that the $n-k$ bodies attract one another and are attracted by each of the k bodies according to Newtonian law. A properly restricted problem is one in which the k bodies are stipulated to move in accordance with a particular solution of the general k-body problem. This paper deals with the properly restricted problems in three bodies for which $k=2$, among which occurs the classical restricted problem of three bodies. The motions stipulated for the two bodies in this paper are the rectilinear solutions of the general two-body problem. If the rectilinear solution be of parabolic type, suitable variables x, y, t may be introduced so that the differential equations for the motion of the third body are $x''+x'=2\Omega_x$, $y''+y'=2\Omega_y$, where Ω is the same as the function occurring in the classical restricted problem of three bodies. This dissipative dynamical system possesses five equilibrium solutions corresponding to the points in the (x, y)-plane where $\Omega_x=\Omega_y=0$. Any positively (negatively) stable motion tends uniformly towards one of these equilibrium solutions as t tends to $+\infty$ $(-\infty)$. The existence and classification of motions both positively and negatively stable is investigated. (Received November 21, 1938.)

44. Venable Martin: *Monotone transformations of non-compact 2-dimensional manifolds.* Preliminary report.

In this paper, the results of Roberts and Steenrod (*Monotone transformations of two-dimensional manifolds*, Annals of Mathematics, (2), vol. 39 (1938), pp. 851–862) are generalized to collections of compact continua on non-compact 2-dimensional manifolds. Theorem 1 of that paper carries over, and Theorem 2 requires only slight modification. A set K is called an A-space if K is a locally connected, locally compact continuum such that each of its maximal cyclic elements is a 2-manifold, and such that if $P_1, P_2, \cdots, P_n, \cdots$ are points of maximal cyclic elements of K different from 2-spheres, P_j and P_k being on different maximal cyclic elements for $j \neq k$, then the set $\sum_{i=1}^{\infty} P_i$ has no limit point. In the analogues of Theorems 3 and 5, a generalized cactoid is replaced by an A-space, and in the analogue of Theorem 5 the identifications necessary may be denumerably infinite, but the points identified will have no limit point. (Received November 21, 1938.)

45. C. N. Moore: *On a new definition for derivatives of non-integral order.*

In the case of derivatives of the nth order (n an integer), it is well known that the expression $[f(x) - C_{n,1}f(x+\Delta x) + C_{n,2}f(x+2\Delta x) - \cdots + (-1)^n f(x+n\Delta x)]/\Delta x$ will tend to $f^{(n)}(x)$ as Δx tends to zero. For a derivative of the rth order (r non-integral) form the expression $[f(x) - C_{r,1}f(x+\Delta x) + C_{r,2}f(x+2\Delta x) - \cdots + (-1)^k C_{r,k}f(x+k\Delta x)]/\Delta x$, where k is the largest integer less than or equal to $1/\Delta x$, and $C_{r,k}$ represents the $(k+1)$st coefficient in the binomial development of $(1+x)^r$. If the expression formed above tends to a limit as Δx tends to zero, that limit is defined as the derivative of rth order. This definition can be shown to agree with earlier definitions in a variety of cases. (Received November 22, 1938.)

46. Marston Morse and G. A. Hedlund: *Symbolic dynamics.* II. *Sturmian sequences.*

Consider symbolic trajectories X of the form $\cdots c_{-2}c_{-1}c_0c_1c_2 \cdots$ in which c_i is either a or b. Let y be an m-block of X. The number of symbols a or b in y will be termed the a-length or b-length, respectively, of y. If the a-lengths (b-lengths) of any two m-blocks with the same m differ by at most one, X is called Sturmian. Let A_n be the a-length of an arbitrary n-block of X. The limit of A_n/n as n becomes infinite exists and will be denoted by α. Sturmian sequences are characterized by the value of α and two other numerical invariants. The types determined are irrational, skew-symmetric, and periodic. A simple mechanical construction of these types is given, the construction being uniquely determined by the three numerical invariants. The trajectories of irrational type are recurrent but not periodic. Let $R(n)$ be the recurrency index of X. The inferior and superior limits of $R(n)$ n as n becomes infinite are studied. These limits are also investigated from a statistical point of view. Sturmian sequences may be used to characterize the distribution of the zeros of solutions of a differential equation of the form $y''+f(x)y=0$, where $f(x)$ is a periodic function of x. (Received November 25, 1938.)

47. John von Neumann and I. J. Schoenberg: *Fourier integrals and metric geometry.* II.

In a previous paper with the same title (abstract 42-9-353), the authors determined all functions $F(t)$ such that the new space $F(E_1)$, arising from the one-dimensional euclidean space E_1 by changing its metric from PQ to $F(PQ)$, be isometrically imbeddable in Hilbert space $\mathfrak{H}$. In a recent paper (Annals of Mathematics, (2), vol. 39 (1938), pp. 811–841), Schoenberg determined all functions $F(t)$ such that the space $F(\mathfrak{H})$ is isometrically imbeddable in $\mathfrak{H}$. In the present paper the similar problem of finding those $F(t)$ such that $F(E_m)$ be imbeddable in $\mathfrak{H}$ is solved. Its solution is based on certain limiting theorems similar to P. Lévy's limit theorem for the characteristic functions in the calculus of probabilities. (Received November 21, 1938.)

48. C. O. Oakley: *On the representation of line segments in the plane by equalities.*

This paper is an extension of both the methods and results of the author and of V. Alaci in the field of semilinear equations. (For a bibliography see the author's paper in Bulletin Scientifique de l'École Polytechnique de Timisoara, vol. 7, nos. 3, 4, pp. 198–199.) In it is developed, by using certain auxiliary curves and the operators "absolute values" and (or) $\int_0^\infty [(\sin \alpha x)/x]dx$, equations of generalized polygons made up of any m line segments whatsoever. (Received November 23, 1938.)

49. Rufus Oldenburger: *Minimal numbers.*

With each symmetric form F of degree p and field K of order p or more there is associated a minimal number $m(F)$ such that F is a linear combination of $m(F)$ pth powers of linear forms, and not such a combination with less than $m(F)$ forms. In the present paper relations are obtained between the minimal numbers of sums and products of forms and the minimal numbers of these forms. These results have applications in the theories of representation and factorability of forms. Thus if a form of degree p splits into a product of linear factors, its minimal number is not greater than p factorial. (Received November 21, 1938.)

50. Oystein Ore: *A remark on the normal decompositions of groups.*

It is shown that in a representation of a group G as the union of normal indecomposable subgroups, the components of nonabelian type are not only normally indecomposable in G, but also in themselves. (Received November 23, 1938.)

51. J. C. Oxtoby: *Note on metrically transitive transformations.* Preliminary report.

In the metric group of all measure-preserving homeomorphisms of the n-dimensional cube, $(n \geqq 2)$, which leave the boundary fixed, the metrically transitive transformations form a residual set. Its Borel type is $G_{\sigma\delta}$. (Received November 22, 1938.)

52. Edmund Pinney: *General geodesic coordinates of order r.* Preliminary report.

This paper is a generalization of a paper entitled *Geodesic coordinates of order r*, by A. D. Michal (this Bulletin, vol. 36 (1930), pp. 541–546). In a geometric space having Banach coordinates and in which open sets satisfying Sierpiński's axioms (i), (ii), (iii), and (v) are defined, contravariant vectors, linear connections of the second and higher orders, and tensors are defined. Geodesic coordinates of order r are defined, and an example is given together with the application of these coordinates to the evaluation of the covariant derivatives of tensors and linear connections. (Received November 26, 1938.)

53. Everett Pitcher: *The critical points of a map on a circle.*

Let R be a manifold of class C^4, and let f be a map of R on the circle S. In terms of the angular coordinate θ, let f be of class C^2. Critical points of the map are differential critical points of the function θ on R as defined by Morse, and type numbers are assigned to critical sets accordingly. (We assume that critical values of θ are a finite set.) Let M_k be the number of critical points of f of type k, and let R_k be the kth connectivity modulo 2 of R. Then there are numbers $\rho_k \geqq 0$, defined as ranks of suitably chosen groups of cycles, such that $M_k - M_{k-1} + \cdots + (-1)^k M_0 \geqq R_k - R_{k-1} + \cdots + (-1)^k R_0 - \rho_k$, where k ranges from 0 to n, the dimension of R. When $k = n$, the inequality is an equality and $\rho_k = 0$. The method depends on the Morse critical point theory and on the Mayer-Vietoris formulas. (Received November 23, 1938.)

54. G. B. Price: *Spaces whose elements are sets.*

This paper contains a study of metric spaces whose elements are sets in other spaces, in particular, in metric spaces, in Banach spaces, and in spaces in which there is an outer Carathéodory measure. The space whose elements are sets in a complete metric space and in which the metric is the Hausdorff metric is complete. The space whose elements are sets in a space with an outer Carathéodory measure can be metrized in such a way that it is complete. Linear operations are studied in the space whose elements are sets in a Banach space; these include generalizations of the convex extension of a set. The results have applications in the theory of integration. (Received November 21, 1938.)

55. W. T. Puckett: *A fixed cyclic element theorem.*

Let M be a locally connected continuum, and let the continuous transformation $T(M) \subset M$ have the property that if E is a cyclic element of M, then no point separates $T(E)$ in M. It is shown that for such a transformation there exists a fixed cyclic element E_0, that is, $T(E_0) \subset E_0$. There exists, therefore, a fixed cyclic element provided either (1) $T(M) \subset M$ is continuous and for each $y \varepsilon T(M)$ the set $T^{-1}(y)$ does not separate a cyclic element, or (2) $T(M) \subset M$ is a homeomorphism (W. L. Ayres, Fundamenta Mathematicae, vol. 16 (1930), pp. 332–336). Another consequence is the well known theorem that the dendrite has a fixed point under any continuous transformation. (Received November 21, 1938.)

56. W. T. Reid: *Isoperimetric problems of Bolza in non-parametric form.*

This paper is concerned with the proof of an effective Lindeberg theorem for non-parametric problems of the calculus of variations. The problem of Bolza with variable end points and isoperimetric side conditions is considered specifically, and sufficient conditions for a strong relative minimum are obtained. It is of interest to note that the Lindeberg theorem here presented involves only the Weierstrass E-function of the problem under consideration. Moreover, the introduction of an equivalent associated problem renders a simplicity to the proof of the principal results of this paper comparable to that for a problem which involves no auxiliary differential equations. (Received November 21, 1938.)

57. R. F. Rinehart: *An interpretation of the inertia of the discriminant matrices of an associative algebra.*

Let A be a linear associative algebra over a field F. It is well known that the nullity of the first discriminant matrix $T_1(A)$ is equal to the order of the radical of A. The nullity, however, is only one of the invariants of $T_1(A)$ under transformations of the basis of A. If F is the real field, the nullity and the signature, or the nullity and the index of inertia, constitute a complete set of invariants of $T_1(A)$. In this paper the interpretation of the invariants of $T_1(A)$ over the real field is completed by finding the significance of a second integer invariant of $T_1(A)$. The following two theorems are proved. (1) The index of inertia of $T_1(A)$ is equal to the number of linearly independent idempotents $b_1, b_2, \cdots, b_k$ in A/N, N being the radical of A, which have the property that b_ib_j is zero, nilpotent, or equal to b_j, for every pair of the set. (2) The index of inertia is equal to the sum of the number in a complete set of primitive idempotents of A and the order of a maximal nilpotent subalgebra of A, diminished by the order of the radical of A. (Received November 7, 1938.)

58. G. E. Schweigert: *Concerning the hyperspace associated with a pointwise periodic transformation.*

If M is a compact locally connected continuum, $T(M)=M$ is pointwise periodic, and the period function is bounded on each cyclic element of M, then each convergent sequence of orbits has an orbit as its limit set. It is accordingly true that the decomposition of M into its orbits generates an interior transformation. (Received November 21, 1938.)

59. W. T. Scott and H. S. Wall: *Properties of analytic functions derived from their corresponding continued fractions.*

There is a 1–1 correspondence between power series $P(x)$ and continued fractions of the form $c_0+K_1^\infty(a_ix^{\alpha_i}/1)$ where α_i is a positive integer (Walter Leighton, this Bulletin, vol. 42 (1936), p. 184). Conditions are obtained in order that (1) a power series shall correspond to a continued fraction with given α_i, (2) the convergents of the continued fraction shall be Padé approximants for $P(x)$, (3) the sum of the nth convergents of continued fractions corresponding to $P(x)$ and $Q(x)$ shall be a Padé approximant for $P(x)+Q(x)$. For a class of continued fractions it is shown that the analytic functions represented have the unit circle for a natural boundary. (Received November 25, 1938.)

60. J. A. Shohat: *On a differential equation for orthogonal polynomials.*

An elementary method is developed for the effective construction of a homogeneous linear differential equation for a certain class of orthogonal polynomials (OP), the coefficients being polynomials of fixed degrees. Application is made to the classical OP, also to new examples. Some new asymptotic relations are derived in the theory of mechanical quadratures. (Received November 14, 1938.)

61. J. A. Shohat and J. D. Tamarkin: *On mechanical quadrature formulas for infinite intervals.*

In the existing literature the question of convergence of mechanical quadrature formulas for infinite intervals has been treated under rather restrictive assumptions. In the present paper, a new method is developed which seeks to unify the previous results and to derive new ones, considerably more general. In particular, the assumption of the uniqueness of the corresponding problem of moments, which figured implicitly in previous investigations, is shown to be unnecessary. (Received November 23, 1938.)

62. F. C. Smith: *Relations among the fundamental solutions of the generalized hypergeometric equation when $p=q+1$. II. Logarithmic cases.*

In a previous paper (this Bulletin, vol. 44 (1938), pp. 429–433), the author obtained the relations between the non-logarithmic solutions of the generalized hypergeometric equation about the point $z=0$ and those about the point $z=\infty$ for the case in which $p=q+1$. In the present paper, similar relations are developed when one or both sets of solutions contain logarithmic members. The results generalize those of Lindelöf (Acta Societatis Scientiarum Fennicae, vol. 19 (1893), pp. 3–31) and Mehlenbacher (American Journal of Mathematics, vol. 60 (1938), pp. 120–128) who treated the case in which $p=2$ and $q=1$. (Received November 14, 1938.)

63. R. H. Sorgenfrey: *A theorem on atriodic continua.*

It has been shown by R. L. Moore (Proceedings of the National Academy of Sciences, vol. 20, pp. 41–45) that if S is a compact, nondegenerate atriodic continuum lying in the plane α, and S contains no continuum which separates α, then S is irreducible between some two of its points. The following generalization has been obtained: Every compact nondegenerate continuum which is atriodic and unicoherent is irreducible between some two of its points. (Received November 29, 1938.)

64. Ruth R. Struik and Miriam van Waters: *A mathematical method applied to criminology.*

This paper is the result of two years' investigation of records in the research department of the Reformatory for Women in Massachusetts. It is an endeavor to reduce life histories of women offenders to certain mathematical symbols and to relate them to the prognosis of success or failure on release from prison. The authors, however, do not attempt to support a hypothesis of predictability of human behavior on the evidence only of the correlations obtained. (Received November 26, 1938.)

65. Alvin Sugar: *On a theorem in additive theory of numbers.*

In this note the author obtains a generalization of an ascension theorem of Dickson's and applies it to several interesting cases. (Received November 23, 1938.)

66. Otto Szász: *On the Cesàro and Riesz means of Fourier series.*

Let a function be positive in the interval $(0, \pi)$ and convex upward. Fejér has shown that its sine series has all Cesàro means of third order positive and convex upward, but that this is not true in general for the Cesàro means of second order. The author shows that the former property holds for Riesz means of second order. Other generalizations of some of Fejér's results are given. (Received November 22, 1938.)

67. C. C. Torrance: *Topologies in which the operation of closure is continuous.*

In this paper a study is begun of abstract spaces S in which the operation of closure is continuous with respect to closure in the space of subsets of S. (Received November 22, 1938.)

68. S. M. Ulam: *On the abstract theory of measure.*

Let $\{A_n\}$, $(n=1, 2, \cdots)$, be a sequence of abstract sets, and let $B\{A_n\}$ be the Borel field over these sets, that is, the smallest class of sets containing the sets A_n and closed with respect to the operations of infinite summation and intersection. Necessary and sufficient conditions are found under which it is possible to define an absolutely additive measure for all sets in $B\{A_n\}$. The same problem is solved when a notion of *congruence* is defined for the sets A_n, and when one postulates equality of measure for congruent sets. (Received November 22, 1938.)

69. A. D. Wallace: *Connected coverings and monotone transformations.*

The results of this paper are based on work by H. Hopf (*Freie Überdeckungen und freie Abbildungen*, Fundamenta Mathematicae, vol. 28 (1937), p. 33). In particular, the author relates finite coverings by connected sets with monotone transformations. Under the assumption that S is a Peano space, five theorems are proved. (1) If S admits a free transformation into a space of dimension at most k, it admits a free monotone transformation into a space of dimension at most k. (2) In order that S admit a free monotone transformation into a space of dimension at most k, it is necessary and sufficient that it admit a free finite covering of order at most $k+1$ with closed and connected sets. (3) S does not admit a free monotone transformation into a dendrite. (4) In order that S be unicoherent, it is necessary and sufficient that every finite covering of order 2 with closed and connected sets have an acyclic nerve. (5) In order that S be unicoherent, it is necessary and sufficient that every 1-dimensional monotone image of S be acyclic. (Received November 19, 1938.)

70. A. D. Wallace: *On interior transformations.*

Assume that S is a Peano space and $T(S)=S_1$ is an interior transformation. In this paper, the following results are obtained: (1) If T is light, then about each point in S there is an arbitrarily small open connected set R such that $T(F(R))=F(T(R))$, where $F(R)=\bar{R}-R$. (2) If X is a closed subset of S_1, then X separates S_1 irreducibly between the points p and q if and only if $T^{-1}(X)$ separates S irreducibly between $T^{-1}(p)$ and $T^{-1}(q)$. If P is a property of point sets, denote by $Q(P)$ the property that the boundary of every region has P. (3) If P is an interior invariant and S has $Q(P)$, so also has S_1. (4) If P is finitely additive and S has $Q(P)$, then $T^{-1}(x)$ has P for each x in S_1. (Received November 19, 1938.)

71. J. L. Walsh: *Maximal convergence of sequences of rational functions.*

Let S be a region bounded by a finite number of disjoint Jordan curves C_0 and a finite number of disjoint Jordan curves C_{-1} disjoint with C_0. Let $u(x, y)$ be harmonic in S, continuous in the corresponding closed region $\bar{S}$, equal to zero and -1 on C_0 and C_{-1}. Denote by C_σ the locus $u(x, y)=\sigma$, $(0>\sigma>-1)$, and by S_σ the region bounded by C_σ and C_{-1}. Let $f(z)$ be analytic throughout S_ρ but not throughout any $S_{\rho'}$, $(\rho'>\rho)$, and let $f(z)$ be continuous on C_{-1}. Let $f_n(z)$ be analytic in S, continuous in $\bar{S}$, with $\limsup_{n\to\infty} [\max|f_n(z)|,\ z \text{ on } C_0]^{1/n} \leqq e^\alpha > 1$, and $\limsup_{n\to\infty} [\max|f(z)-f_n(z)|,\ z \text{ on } C_{-1}]^{1/n} \leqq e^\beta < 1$. Then $\alpha+\alpha\rho-\beta\rho \geqq 0$; and the equality sign here implies $\limsup_{n\to\infty} [\max|f_n(z)|,\ z \text{ on } C_\mu]^{1/n} = e^{(\alpha-\beta)(\mu-\rho)}$, $(0\geqq\mu\geqq\rho)$, $\limsup_{n\to\infty} [\max|f(z)-f_n(z)|,\ z \text{ on } C_\sigma]^{1/n} = e^{(\alpha-\beta)(\sigma-\rho)}$, $(\rho>\sigma\geqq-1)$. In this theorem (latter part) are included many of the expansions in series of rational functions recently studied by the writer (*Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 20 (1935), chaps. 8 and 9). The concept and properties of maximal convergence of expansions in polynomials extend thereby to expansions in rational functions whose prescribed poles satisfy asymptotic conditions. (Received November 19, 1938.)

72. J. L. Walsh: *Note on the location of zeros of the derivative of a rational function whose zeros and poles are symmetric in a circle.*

Let the points α_k be given interior to C: $|z|=1$, and let Π be the smallest closed curvilinear polygon interior to C bounded by arcs of circles Γ orthogonal to C, such that Π contains each of the points α_k, and such that each Γ bounds a region containing in its interior no α_k. Then Π contains on or within it all the zeros of $r'(z)$, where $r(z) \equiv \lambda \prod_{k=1}^{k=m} [(z-\alpha_k)/(1-\bar{\alpha}_k z)]$, $(|\lambda| = 1)$. This theorem may be considered as the non-euclidean analogue of the Lucas theorem concerning the zeros of the derivative of a polynomial. The theorem extends to Blaschke products and various harmonic functions. (Received November 10, 1938.)

73. S. E. Warschawski: *On functions analytic in a half-plane.*

In 1935 Hille and Tamarkin proved that a function $f(z)$, $z=x+iy$, regular in $x>0$ of the class $\int_{-\infty}^{\infty}|f(x+iy)|^p dy \leq M$ for all $x>0$, $(p \geq 1)$, can be written as $f(z)=b(z)g(z)$, where $b(z)$ is the "Blaschke product" associated with $f(z)$ and $g(z) \neq 0$ and is of the same class as $f(z)$. In the present paper, analogous theorems are proved for the following three classes: (a) $\int_{-\infty}^{\infty}\{(\log^+|f(z)|)/|1+z|^2\}dy \leq M$, (b) $\int_{-\infty}^{\infty} \log^+|f(z)|dy \leq M$, (c) $(1/2)a\int_{-a}^{a}|f(z)|^p dy \leq M$, $(p>0)$, for all $x>0$. Moreover, the first two classes are identical with classes of functions $f(z)$ decomposed in the above manner, for which $\log g(z)$ is representable by certain types of Poisson-Stieltjes integrals. For $p \geq 1$, the functions of the class (c) are representable by the Poisson integral in $x>0$. (Received November 23, 1938.)

74. G. T. Whyburn: *On irreducibility of transformations.*

For a continuous transformation $T(A)=B$ on a compact set A, certain associated "diameter functions" are defined and use is made of the continuity properties of these functions in a study of properties of T. For example, it is shown that T is strongly irreducible (that is, no proper closed subset of A maps onto all of B) if and only if the set E of points x with $x=T^{-1}T(x)$ is dense in A. If A is connected and locally connected, T is irreducible (that is, no proper subcontinuum of A maps onto B) if and only if the set E is dense in the set of non-cut points of A. In either of these cases, if T is interior, it is necessarily a homeomorphism. In general, the set of points on which T is light, that is, the set of points x such that $T^{-1}T(x)$ is 0-dimensional, is a G_δ. (Received November 18, 1938.)

75. G. T. Whyburn: *Semi-locally connected sets.*

A metric connected set M is said to be semi-locally connected if for each $x \in M$ and each $\epsilon>0$ a $\delta>0$ exists such that $M-V_\delta(x)$ is contained in a finite number of components of $M-V_\epsilon(x)$. The class of compact semi-locally connected continua includes but is not included in the class of compact locally connected continua. In this paper a study is made of the structure of such continua, and it is found that they have many of the properties of locally connected continua. For example, each component of a set of the form $M-x$, $x \in M$, is open and strongly connected. The simple links and the 0th order cyclic elements are identical and are themselves semi-locally connected. The intersection of any connected set and one of these elements is either connected or vacuous, each component of the complement of such an element is bounded by a single point, the nondegenerate elements form at most a null sequence, for any two points a and b of M, the sum of a, b and all points separating a and b is a compact set, and so on. (Received November 21, 1938.)

76. E. F. Beckenbach and Maxwell Reade: *A characterization of isothermic spherical maps.*

Continuing the work reported in abstracts 44-3-92 and 44-7-309, the authors prove the following theorem: If $x_j(u, v)$, $(j=1, 2, 3)$, are of class C_3 (that is, all derivatives of order less than or equal to three are continuous) in a simply connected domain D, then a necessary and sufficient condition that these functions map D isothermically on a spherical surface, such that circles are not mapped on circles, is that (1) $\sum_{j=1}^{3}[\int_C x_j(u, v)(du+idv)]^2=o(r^\alpha)$, where C is a circle of radius r in D, hold identically in D for $\alpha=6$ but not for $\alpha=8$. Condition (1) holds identically in D for $\alpha=8$ if and only if it holds for $\alpha=\infty$, and if and only if D is mapped isothermically either on (2) a spherical surface, such that circles are mapped on circles, or on (3) a minimal surface; (2) was omitted from a previous report (44-7-309) on minimal surfaces. (Received December 5, 1938.)

77. Garrett Birkhoff: *The mean ergodic theorem.*

A one-page proof is given of von Neumann's mean ergodic theorem, in the following generalized form: Let X be any uniformly convex Banach space, and T any linear operator on X of norm at most unity. Then the means of any point of X, under transformation by powers of T, converge strongly to a limit. (Received December 28, 1938.)

78. Garrett Birkhoff and Morgan Ward: *A characterization of Boolean algebras.*

It is shown that a lattice of finite dimensions is a Boolean algebra if and only if every element of the lattice has a unique complement. More generally, this holds for complete, atomistic lattices. (Received December 10, 1938.)

79. Harald Bohr and D. A. Flanders: *Algebraic functions of analytic almost periodic functions.*

In a previous paper (Danske Videnskabernes Selskab, Mathematisk-fysiske Meddelelser, vol. 15 (1937), pp. 1–40) the authors have studied the solutions of algebraic equations whose coefficients are almost periodic functions of a real variable. The present paper considers the corresponding problem when the coefficients are analytic almost periodic functions. The basic theorem on the almost-periodicity of the solutions follows readily from the corresponding theorem (due to Walther and Campbell) for the real variable case. It is then shown that if the Dirichlet exponents of the coefficients are bounded below while those of the discriminant have a minimum, the Dirichlet exponents of the solutions are bounded below. Using a theorem of Ostrowski (Mathematische Zeitschrift, vol. 37 (1933), pp. 98–133) concerning the formal solution of algebraic equations whose coefficients are formal ordinary Dirichlet series (that is, with exponents $\lambda_n \uparrow \infty$), it is shown that if the Dirichlet series of the coefficients are of this type, so also are those of the solutions. These results are related to an earlier work of Ritt (Transactions of this Society, vol. 31 (1929), pp. 654–679). (Received December 7, 1938.)

80. O. K. Bower: *Paradoxes involving mathematical expectation.*

If the set of discrete values of a stochastic variable x is e_i, $(i=1, 2, \cdots)$, with probabilities $p(e_i)$, the ordinary definition of expectation associated with the set on a single trial is $p(e_i)M_i$, in which M_i, $(i=1, 2, \cdots)$, represents the amount of gain for the respective values or events. This definition makes expectation zero for games for which it is reasonable to take it otherwise. The present paper redefines expectation so that if the event e_i is imbedded in a finite number of trials, expectation may be taken different from zero; the modified definition coincides with the usual one if the event is imbedded in an infinite number of trials. Application is made to some paradoxes involving mathematical expectation, including the famous St. Petersburg paradox and the martingale. (Received December 19, 1938.)

81. O. K. Bower: *The equation $f(x+y)=f(x)f(y)$ and a system of functional equations depending upon its solution.*

In the discussion of the single equation, the author uses results of an unpublished paper by O. K. Bower and J. D. Grant, *A system of simultaneous bilinear functional equations*, an abstract of which appeared in this Bulletin (abstract 38-7-160). A synopsis of this unpublished paper is given in the present paper. (Received December 22, 1938.)

82. George Comenetz: *The limit of the ratio of arc to chord for a space curve.*

Consider a regular analytic curve in complex euclidean space of three dimensions which passes through a point P in a minimal direction. Let n be the order of contact of the curve with its minimal tangent line at P, and let m be the order of contact of the curve with the minimal plane which contains that minimal line. Let the value at P of the derivative of the radius of curvature of the curve with respect to arc length be c. Then if $m>2n$, the limit of the ratio of arc to chord is 1; if $m<2n$, the limit is $2(m+1)^{1/2}/(m+2)$; and if $m=2n$, the limit is $c\{c^2-(n+1)^2/(2n+1)\}^{-1/2}$. The study of this feature of the geometry of complex curves which are isotropic at an isolated point only was initiated by Kasner (see references in Proceedings of the National Academy of Sciences, vol. 18 (1932), p. 267). (Received December 5, 1938.)

83. J. J. DeCicco: *The analogue of the Moebius group of circular transformations in the associated Kasner plane.*

This is a continuation of two papers by Kasner (Science, vol. 85 (1937), pp. 480–482; Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 337–341), abstract 42-11-398 by Kasner, and abstract 44-11-444 by the author. A simple horn-set is the totality of all curves (third order elements) which possess a common point and direction. Let $x=\gamma$, $y=d\gamma/ds$, where γ is the curvature and s is the arc length. A simple horn-set is then called the associated Kasner plane K_2, where any point of K_2 is a curve (x, y) of the simple horn-set. By the group of conformal transformations, one finds that two distinct points of K_2 possess the unique conformal measure $(x_2-x_1)^2/(y_2-y_1)$. The locus of points of K_2 at a constant conformal measure from a fixed point is called a parabolic circle. In this paper is found the group of point transformations in K_2 which convert parabolic circles into parabolic circles. The result is a continuous seven-parameter group G_7, the parabolic circle group. Moreover, if a point transformation of K_2 converts more than $2\infty^2$ parabolic circles into parabolic circles, then it must be a parabolic circle transformation. (Received December 19, 1938.)

84. J. J. DeCicco: *The polygenic functions whose associated series are equiparallel series.*

This is a continuation of two papers by Kasner, *A complete characterization of the derivative of a polygenic function* (Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 172–177) and *The polygenic functions whose associated element-to-*

point transformations convert unions into points (this Bulletin, vol. 44 (1938), pp. 726–733). Kasner has shown that the derivative $\gamma = dw/dz$ of a polygenic function $w = \phi(x, y) + i\psi(x, y)$ with respect to $z = x + iy$ defines an element-to-point transformation T. This transformation is characterized by the three properties: (1) the circle property; (2) the ratio $-2:1$ property; (3) the affine-similitude property. The ∞^2 series of the γ-plane which by T are converted into the ∞^2 points of the z-plane are called the associated series of w. In the present paper all polygenic functions whose associated series are equiparallel series are obtained explicitly. There are three distinct fundamental types (A), (B), and (C) of polygenic functions with this property. The corresponding geometric situations in the z- and γ-planes under each of these three fundamental types (A), (B), and (C) are also obtained. (Received December 19, 1938.)

85. Nelson Dunford and B. J. Pettis: *On completely continuous operations in L.*

If L is the space of functions defined and integrable over the unit interval, then an operation $T(\phi) = \psi$, such that to each ϕ in L is assigned an element ψ in L, is linear and completely continuous if and only if $T(\phi)$ can be represented as $T(\phi) = \int_0^1 K(s, t)\phi(t)dt$, where (1) $K(s, t)$ is measurable and vanishes outside of the unit square, (2) ess.sup. $\int_0^1 |K(s, t)| ds \equiv C < \infty$, and (3) $\lim_{h \to 0}$ ess.sup. $\int_0^1 |K(s+h, t) - K(s, t)| ds = 0$. The norm of the operation is the constant C in (2). The theorem has applications to integral equations. (Received November 22, 1938.)

86. H. L. Garabedian: *A sufficient condition for Cesàro summability.*

It is proved in this paper that if $\Delta^{k-1}a_0 \neq 0$, $\Delta^i a_0 = 0$, $(i \geqq k \geqq 1)$, then the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is exactly summable (C, k) to the value $\sum_{n=0}^{k-1}\Delta^n a_0/2^{n+1}$. It is known that summability $(E, 1)$, Euler summability of order one, is consistent with summability (C, k). However, neither method of summability includes the other. The result of this paper exhibits a class of series summable by both methods. (Received December 10, 1938.)

87. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms in the plane.* Preliminary report.

This paper considers the following problem of W. L. Ayres. Does there exist in the plane a continuous curve M which is not disconnected by the removal of any pair of its points and a pointwise periodic homeomorphism $Y(M) \equiv M$ with the following properties: (a) M has three fixed points (at least) under Y; (b) M has a point of period greater than two under Y? In this preliminary note it is shown that if Y is periodic, the answer to the above problem is in the negative. (Received December 27, 1938.)

88. L. B. Hedge: *Integration in a compact metric space.*

In a metric space which is either compact or expressible as a sum of a countable number of compact subspaces, the behavior of certain set functions, defined over open sets, is investigated. An important class of sets, regions of continuity, is defined for these functions of open sets. This class of sets is characterized and made the basis for a system of integration, and several convergence theorems are proved. (Received December 5, 1938.)

89. Einar Hille: *Remarks concerning group spaces and vector spaces.*

This note gives a brief discussion of abstract spaces of two different types generalizing the types (G) and (F) of Banach. In the first case, here called a group space, addition is the basic operation; in the second, here called a continuous vector space, scalar multiplication is also defined. The topology is based upon closure rather than upon distance, and in order that the basic operations be continuous, it is postulated that the notion of closure be invariant under the basic operations. Various applications of these definitions are given. (Received January 3, 1939.)

90. R. P. Isaacs: *A geometric interpretation of the difference quotient of polygenic functions.*

Let $f(z)$ be any polygenic function and $\omega=\rho e^{i\theta}$. Define the operators $\Delta f(z)=\omega^{-1}[f(z+\omega)-f(z)]$ and $\mathcal{D}=e^{i\theta}\partial/\partial z+e^{-i\theta}\partial/\partial\bar{z}$. Then $\Delta=\omega^{-1}[e^{\rho D}-1]$ or (1) $\gamma=\Delta f(z)=A_{10}+A_{11}e^{-2i\theta}+\rho(A_{20}e^{i\theta}+A_{21}e^{-i\theta}+A_{22}e^{-3i\theta})+\rho^2(A_{30}e^{2i\theta}+\cdots+A_{33}e^{-4i\theta})+\cdots$, where the A_{ij} are complex numbers for any particular z. If (1) is plotted as a surface for some particular z, with the rectangular coordinates $R(\gamma)$, $I(\gamma)$, and ρ, and with θ as a parameter, an hourglass shaped figure is obtained. The derivative of f at the section $\rho=0$ shows that this section, the "waste" of the hourglass, is the Kasner derivative circle. The surface is therefore called the Kasner-coid or K-coid. A geometrical interpretation of the difference quotient of a polygenic function is thus obtained. (Received December 13, 1938.)

91. Nathan Jacobson: *Cayley numbers and Lie algebras of type G.*

This paper determines all simple Lie algebras of type G (order 14) over any field of characteristic 0. These algebras are represented as the algebras of derivations of generalized Cayley algebras. The Lie algebras are isomorphic if and only if the corresponding Cayley systems are. The group of automorphisms of a Cayley algebra and the group of automorphisms of its derivation algebra are isomorphic. The determination of Cayley systems may be reduced, as is known, to a problem on equivalence of certain quadratic forms in 8 variables. (Received December 5, 1938.)

92. B. O. Koopman: *The axioms and algebra of intuitive probability.*

Probability, in its original sense, belongs to the category of intuition. Between this and the numerical probability of mathematicians there has long subsisted a logical gap. It is the object of the present paper to attempt to bridge this by implementing the concept of intuitive probability with a precise formulation based on a system of axioms, and thence to derive the numerical probability together with all its classical properties. The mathematical groundwork is a Boolean algebra of (contemplated) propositions; the notion of introducing various bodies of specific information has its counterpart in the modular reduction of this algebra to quotient algebras with respect to its various ideals; and the intuition of probability is given its mathematical expression in a partial ordering relation between the diverse remainder classes in the Boolean algebra. Thirteen axioms are then introduced on an a priori intuitive basis, and a systematic theory is developed from them, leading eventually to the classical theory. It is worth remarking that no use of the whole paraphernalia of "collectives" and "multiple-valued truth functions" has been made. (Received December 16, 1938.)

93. E. H. Larguier: *On the foundations of mathematics.*

This paper is a discussion of the origin and the nature of the concepts used in mathematics. (Received December 16, 1938.)

94. L. B. Robinson: *On an equation solved by a fonctionnelle.*

For the equation (1) $\lambda(1+x^m)u'(x)=x^pu(x^2)-kx^r$, where m, p, r are positive integers, successive approximations give a singular solution holomorphic except at zero and infinity. The formula for the nth term of this solution can be calculated. To prove convergence assume that λ is not too great. If $k=0$, a new equation is obtained. By making use of the result given above, a secondary solution with the following characteristics can be found: (1) It has singular points at zero and infinity; (2) it admits the circumference of the unit circle as a singular line; (3) it depends on an infinite number of parameters and is therefore a "fonctionnelle." (Received December 20, 1938.)

95. Barkley Rosser: *Definition by induction in Quine's "New foundations for mathematical logic."*

In Quine's *New foundations* (W. V. Quine, *New foundations for mathematical logic*, American Mathematical Monthly, vol. 44 (1937), pp. 70–80) the axiom of infinity does not appear to be provable. In a certain stronger system, very closely related to Quine's *New foundations*, the axiom of infinity is provable. One of the peculiarities of this latter system is that even unstratified propositions can be proved by induction (this is used in the proof of the axiom of infinity). It would seem that definition by induction should be possible quite irrespective of any conditions of stratification in this latter system. In this paper it is shown that such is the case. (Received December 24, 1938.)

96. Barkley Rosser: *On the first case of Fermat's last theorem.*

In this paper it is proved that if $x^p+y^p+z^p=0$ has a solution with x, y, and z integers in the field of the pth roots of unity and prime to p, then $p\geqq 8{,}332{,}403$. The method of proof is as follows. If $m^{p-1}\equiv 1 \pmod{p^2}$, then m is the pth power of a primitive root modulo p^2. There are $p-1$ such pth powers, and (since -1 is a pth power) half of them occur between 0 and $p^2/2$, and half between $p^2/2$ and p^2. So $m^{p-1}\equiv 1 \pmod{p^2}$ for exactly $(p-1)/2$ m's between 0 and $p^2/2$. However, if there is such a solution of the Fermat equation as described above, then, by a theorem of Morishima, $m^{p-1}\equiv 1 \pmod{p^2}$ for each prime $m\leqq 31$, and hence for any m having no prime factor greater than 31. It is then shown that, for all x's less than or equal to 8,332,366, the number of m's less than $x^2/2$ and having no prime factor greater than 31 is greater than $x/2$, and this proves the theorem. (Received December 24, 1938.)

97. Vivian E. Spencer: *Extensions of theorems of Markoff and Krein.*

These theorems deal with the behavior of the zeros $\{x_{in}\}$ of sequences $\{\Phi_n(x)\}$ of orthogonal polynomials as functions of the associated sequences of moments $\{\alpha_n\}$ and of the sequences of constants $\{c_n\}$ and $\{\lambda_n\}$ determining the corresponding recurrence relation. The extended results concern x_{in} as a function of $\{\alpha_i\}_0^{2n-1}$. Markoff's original theorem is restricted to polynomials all of whose zeros x_{in} are positive. This restriction has been removed, and special cases have been considered. One of the latter is found also to be a consequence of the extended Krein theorem. (Received December 20, 1938.)

98. R. L. Swain: *On continua obtained from sequences of simple chains of connected regions.*

Suppose that $G_1, G_2, G_3, \cdots$ is a sequence of collections satisfying R. L. Moore's Axiom 1 (R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13). Suppose that for each positive integer n, C_n is a simple chain of connected regions of G_n such that C_n^* contains the closure of C_{n+1}^*, where C_n^* is the sum of the point sets of the chain C_n. Then the common part of the point sets $C_1^*, C_2^*, C_3^*, \cdots$ is a compact atriodic continuum, every subcontinuum of which is unicoherent. (Received December 27, 1938.)

99. Annette Vassell: *Sectional families of curves.*

Let S be an arbitrary surface. A sectional family is a triply infinite family of plane curves obtained by projecting centrally upon a fixed plane all the plane sections of S. This paper completes the solution of the problem first considered by Kasner in 1908 of finding a characteristic set of geometrical properties of sectional families (see references in his paper, *Dynamical trajectories and the ∞^3 plane sections of a surface*, Proceedings of the National Academy of Sciences, vol. 17 (1931), p. 370). A characteristic set of seven properties is found for families derived from a general nondevelopable surface S, and a different set of seven properties is found for developables. The work is analogous to Kasner's differential geometric characterization of dynamical trajectories (Transactions of this Society, vol. 7 (1906), p. 401). (Received December 5, 1938.)

100. Morgan Ward: *A characterization of Dedekind structures.*

Let Σ be a structure (lattice) in which a chain condition holds. It is proved that if for every pair of elements a and b of Σ the quotient structures $[a, b]/a$ and $b/(a, b)$ are isomorphic, then Σ is a Dedekind structure (modular lattice). (Received December 24, 1938.)

101. Morgan Ward: *A note on divisibility sequences.*

A sufficient condition is given that the binomial coefficients belonging to any sequence of nonvanishing rational integers may be rational integers. The result is applied to prove that the binomial coefficients belonging to the Lucasian sequences recently studied by the author (Transactions of this Society, vol. 44 (1938), pp. 68–86) are usually integral. (Received December 24, 1938.)

102. Morgan Ward: *Finite point lattices.*

An element p of a lattice Σ with a null element z is said to be a point if it covers z. We call Σ a point lattice if every element in it is a union of points. It is proved that every finite point lattice may be obtained from a finite Boolean algebra by defining a suitable equivalence relation over it. The elements of the point lattice are the equivalence classes of the relation. The correspondence between the Boolean algebra and the point lattice thus induced preserves division and crosscut, but not union. It is also shown that every finite lattice may be imbedded in a point lattice, and conditions on the equivalence relation are given so that the point lattice may be modular. (Received December 24, 1938.)

103. Morgan Ward: *Note on the general rational solution of the equation $ax^2-by^2=cz^3$.*

All rational solutions of the equation $a_0x^m+a_1x^{m-1}y+\cdots+a_my^m=z^n$, a_i rational, are obtained under the assumption that m and n are coprime integers, and the result is applied to find all rational solutions of $ax^2-by^2=z^3$. The procedure is unfortunately not applicable to finding integral solutions of the equations. (Received December 24, 1938.)

104. Morgan Ward: *Ring homomorphisms which are also lattice homomorphisms.*

Let O be a commutative ring with unit element. Then if every ideal of O is principal, it is shown that O is a residuated distributive lattice with respect to the usual division relation in which the ascending chain conditions hold. It is shown that every ring homomorphism of O is also a lattice homomorphism to the sublattice of all divisors of the element m where (m) is the principal ideal fixing the ring homomorphism. The homomorphism theory of principal ideal rings (in which O is a domain of integrity) and finite Boolean rings (Boolean algebras) are included as special cases. (Received December 24, 1938.)

105. Morgan Ward: *The algebra of lattice functions.*

E. T. Bell's theory of numerical functions on the positive integers to the complex numbers is generalized to functions on a quotient set of a semi-ordered set to a division algebra, by a suitably defined "Dirichlet multiplication." The theory of factorable numerical functions and multiplicative numerical functions is extended to any lattice, and a detailed study is made of a generalized norm on the lattice to an abelian group. The results of Glivenko on distance functions in a lattice and of Dedekind on module symbols over a lattice are shown to be intimately connected. Incidentally, a new characterization of a modular nondistributive lattice is given. Let γ be such a lattice with the property that if $a\supset b$ in γ and $a\neq b$, there exists an element c such that a covers c and $c\supset b$. Then γ must contain a modular sublattice of order five which is "complete" in γ. A lattice γ' is said to be complete in a superlattice γ if a covers b in γ' if and only if a covers b in γ. (Received December 24, 1938.)

106. Morgan Ward and R. P. Dilworth: *Evaluations over residuated structures.*

The positive reals and zero form a structure (lattice) with respect to the relation "less than or equal to." An evaluation of a residuated structure Σ is a homomorphism to a structure of positive reals with the additional property that the product of two elements in the structure corresponds to the sum of the corresponding pair of real numbers. Evaluations are shown to be of frequent occurrence. The authors study chiefly discrete evaluations in which the structure of reals consists of the integers $0, 1, 2, \cdots$. The equivalence classes of the evaluation are shown to be substructures whose elements form a descending chain of primary elements. Evaluations which preserve residuation as well as multiplication are also studied, but are of less immediate interest. (Received December 24, 1938.)

107. Morgan Ward and R. P. Dilworth: *Lattice theory of ova.*

The theory of residuated lattices developed by the authors (Proceedings of the National Academy of Sciences, vol. 24 (1938), pp. 162–164) is applied to the problem of imbedding an ovum in an arithmetic recently treated by A. H. Clifford and others (Annals of Mathematics, (2), vol. 39 (1938), pp. 594–610). An ovum O is a set of objects satisfying all the postulates for an abelian group save the existence of inverses. Two classes of distinguished subsets of O are studied, product ideals and ovoid ideals, both of which form completely closed residuated lattices. A concise characterization of the possible arithmetical behavior of the ovum is thereby obtained. In particular, it is shown that by the adjunction of a finite number of ideals, every finite ovum may be imbedded in a residuated lattice in which every element, and in particular the elements of the ovum, may be uniquely represented as a crosscut of primary elements. (Received December 24, 1938.)

108. Kurt Gödel: *The consistency of the generalized continuum-hypothesis.*

We use the following definitions: 1. $M_0=\Lambda$; 2. $M_{\alpha+1}$ is a set of those subsets of M_α which can be defined by propositional functions containing only the following concepts: $\sim$, $\vee$, the ϵ-relation, elements of M_α, and quantifiers for variables with range M_α; 3. $M_\beta=\sum_{\alpha\subset\beta}M_\alpha$ for limit numbers β. Then M_{ω_ω} or M_Ω (Ω being the first inaccessible number) is a model for the system of axioms of set theory (as formulated by A. Fraenkel, J. von Neumann, T. Skolem, P. Bernays) respectively without (or with) the axiom of substitution, the generalized continuum-hypothesis ($2^{\aleph_\alpha}=\aleph_{\alpha+1}$) being true in both models. Since the construction of the models can be formalized in the respective systems of set theory themselves, it follows that $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ is consistent with the axioms of set theory, if these axioms are consistent with themselves. The proof is based on the following lemma. Any subset of M_{ω_α}, which is an element of some M_β, is an element of $M_{\omega_{\alpha+1}}$. This lemma is proved by a generalization of Skolem's method for constructing enumerable models. Since the axiom of choice is not used in the construction of the models, but holds in the models, the consistency of the axiom of choice is obtained as an incidental result. (Received December 29, 1938.)