ON SERRET'S INTEGRAL FORMULA*

G. E. RAYNOR

In 1901 F. Morley† evaluated the integral

$$(1) \qquad \int_0^{\pi/2} \phi^n \log^m (2 \cos \phi) d\phi$$

for even nonnegative integral values of n, and m a positive integer. His method was that of contour integration. However, the results could have been obtained with the help of a formula derived by A. R. Forsyth a few years earlier‡ and also from a formula derived by Cauchy as early as 1825.§ Cauchy's formula is

$$\int_0^{\pi/2} \cos^p \phi \cos q\phi d\phi = \frac{\pi}{2^{p+1}} \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)}, \qquad R(p) > -1.$$

Here p and q are in general complex numbers with p subject to the restriction indicated. If this relation be written in the form

$$\int_0^{\pi/2} (2 \cos \phi)^p \cos q\phi d\phi = \frac{\pi}{2} \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)}$$

and then differentiated m times with respect to p and n times (n even) with respect to q, and then p and q set equal to zero, the values of integrals of the form (1) are obtained quite easily.

In 1843 Serret|| obtained the formula

$$(2) \qquad \begin{aligned} &\int_0^{\pi/2} \cos^p \phi \sin q\phi d\phi \\ &= \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)} \int_0^1 \frac{t^{(p-q)/2} - t^{(p+q)/2}}{(1+t)^{p+1}(1-t)} dt, \\ &\qquad R(p) > -1. \end{aligned}$$

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† This Bulletin, vol. 7 (1901), p. 390.

‡ A. R. Forsyth, *Evaluation of two definite integrals*, Quarterly Journal of Mathematics, vol. 27 (1895), p. 221.

§ A. Cauchy, *Mémoire sur les Intégrales Définies Prises entre des Limites Imaginaires*, Paris, 1825, p. 40.

|| Alfred Serret, Journal de Mathématiques Pures et Appliquées, vol. 8 (1845), p. 7. The formula as given in Serret's article contains two misprints which are repeated in Encyklopädie der mathematischen Wissenschaften, vol. 2, part 2, no. A12, p. 853. They are corrected below.

This result has been quite neglected in the literature on definite integrals. It is the purpose of this note to show how this formula, together with certain well known results of gamma function theory, may be used to study an important class of definite integrals and incidentally to *supplement* certain recent results.

In the following we have employed a number of improper integrals and have also repeatedly differentiated under the integral sign. It is, of course, necessary to make sure that all the integrals used exist, and to justify the double limit processes. This the writer has done, but the details have been omitted in the belief that the reader can easily supply them for himself.

In two interesting papers* Rutledge and Douglass have treated a number of definite integrals which are related to those of the type (1) with *n odd* and to the integral $\int_0^1[(\log u)/u] \log^2 (1+u)du$ appearing in the title of their first paper. Two of their principal results are embodied in the following formulas:

$$\int_0^\pi \phi \log^2\left(2 \cos \frac{\phi}{2}\right) d\phi = A_4 + \frac{31\pi^4}{480}, \tag{3}$$

$$\int_0^1 \frac{\log u}{u} \log^2 (1 + u)du = A_4 - \frac{\pi^4}{288} \cdot \tag{4}$$

The constant A_4 is defined by the equation

$$A_4 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right). \tag{5}$$

By an obvious transformation equation (3) is seen to be equivalent to

$$\int_0^{\pi/2} \phi \log^2 (2 \cos \phi)d\phi = \frac{A_4}{4} + \frac{31\pi^4}{1920} \cdot \tag{6}$$

With the help of expansions in series, contour integration and algebraic combinations various integrals of the nature of that in (4) have been evaluated.† We wish to show how Serret's formula may be used to supplement this work and, as an example, shall evaluate an integral algebraically independent of those given in the table of integrals of order four by Rutledge and Douglass.†

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* 1. George Rutledge and R. D. Douglass, *Evaluation of* $\int_0^1[(\log u)/u] \log^2 (1+u)du$ *and related definite integrals*, American Mathematical Monthly, vol. 41 (1934), pp. 29–36. 2. *Table of definite integrals*, American Mathematical Monthly, vol. 45 (1938), pp. 525–530.

† G. Rutledge and R. D. Douglass, loc. cit. (2), p. 527.

We set down here some formulas needed later and which are to be found in Nielsen's book on the gamma function:*

$$(7)\qquad \Psi(1+x) = -s_1 + s_2x - s_3x^2 + s_4x^3 - s_5x^4 + \cdots;$$

$$(8)\qquad \beta(1+x) = \sigma_1 - \sigma_2 x + \sigma_3 x^2 - \sigma_4 x^3 + \sigma_5 x^4 - \cdots;$$

$$(9)\qquad s_1 = C = \text{Euler's constant}; \qquad s_n = \sum_{p=1}^{\infty} \frac{1}{p^n}, \qquad n > 1;$$

$$(10)\qquad \sigma_n = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^n}, \qquad n \geqq 1;$$

$$(11)\qquad \sigma_1 = \log 2, \qquad \sigma_n = (1 - 1/2^{n-1})s_n, \qquad n > 1;$$

$$(12)\qquad s_{2n} = \frac{B_n(2\pi)^{2n}}{(2n)!2}, \qquad B_1 = \frac{1}{6}, \qquad B_2 = \frac{1}{30};$$

$$(13)\qquad \beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \qquad R(x) > 0;$$

$$(14)\qquad \Psi(x) + C = \int_0^1 \frac{1-t^{x-1}}{1-t}\,dt, \qquad R(x) > 0;$$

$$(15)\qquad \eta(x) = \int_0^1 \frac{\log 2 - \log(1+t)}{1-t}\,t^{x-1}dt, \qquad R(x) > 0;$$

$$(16)\qquad \beta^2(x) = \Psi^{(1)}(x) - 2\eta(x).$$

The notation $\Psi^{(n)}(x)$ means the nth derivative of $\Psi(x)$.

Now let each side of (2) be differentiated with respect to q and then q set equal to zero. We obtain

$$(17)\qquad \int_0^{\pi/2} \phi \cos^p \phi d\phi = -\frac{\Gamma(p+1)}{\Gamma^2(p/2+1)} \int_0^1 \frac{t^{p/2}}{(1+t)^{p+1}(1-t)} \log t dt.$$

Set

$$(18)\qquad J(p) = \Gamma(p+1)/\Gamma^2(p/2+1).$$

Then since, by definition, $\Psi(x) = \Gamma^{(1)}(x)/\Gamma(x)$, it follows with the help of (7), that

$$(19)\qquad J^{(1)}(0) = 0, \qquad J^{(2)}(0) = s_2/2.$$

Also let

$$(20)\qquad T(p) = (t^{1/2}/(1+t))^p.$$

Then

* N. Nielsen, *Handbuch der Theorie der Gammafunktion*, equations (7), (8), (9), (10), (11), p. 37; (12), p. 46; (13), p. 169; (14), p. 170; (15), p. 195; (16), p. 194.

$$T^{(n)}(p) = (t^{1/2}/(1+t))^p \log^n t^{1/2}/(1+t) \tag{21}$$
$$= (t^{1/2}/(1+t))^p[(\log t)/2 - \log(1+t)]^n.$$

Now differentiating (17) twice with respect to p and then letting $p=0$, we have by (19) and (21)

$$\int_0^{\pi/2} \phi \log^2 \cos \phi d\phi = -\frac{1}{2} s_2 \int_0^1 \frac{\log t dt}{(1+t)(1-t)} \tag{22}$$
$$- \int_0^1 \frac{[(\log t)/2 - \log(1+t)]^2 \log t}{(1+t)(1-t)} dt.$$

From (17), for $p=0$,

$$\int_0^{\pi/2} \phi d\phi = -\int_0^1 \frac{\log t dt}{(1+t)(1-t)},$$

or

$$\frac{\pi^2}{8} = -\int_0^1 \frac{\log t dt}{(1+t)(1-t)}. \tag{23}$$

Hence (22) becomes, since by (12) $s_2 = \pi^2/6$ and $s_4 = \pi^4/90$,

$$\int_0^{\pi/2} \phi \log^2 \cos \phi d\phi$$
$$= \frac{15}{16} s_4 - \int_0^1 \frac{[(\log t)/2 - \log(1+t)]^2 \log t}{(1+t)(1-t)} dt$$
$$= \frac{15}{16} s_4 - \frac{1}{8}\int_0^1 \frac{\log^3 t}{1+t} dt - \frac{1}{8}\int_0^1 \frac{\log^3 t}{1-t} dt \tag{24}$$
$$+ \frac{1}{2}\int_0^1 \frac{\log(1+t)\log^2 t}{1+t} dt + \frac{1}{2}\int_0^1 \frac{\log(1+t)\log^2 t}{1-t} dt$$
$$- \frac{1}{2}\int_0^1 \frac{\log^2(1+t)\log t}{1+t} dt - \frac{1}{2}\int_0^1 \frac{\log^2(1+t)\log t}{1-t} dt.$$

Now from (13), (8) and (11) we have

$$\int_0^1 \frac{\log^3 t}{1+t} dt = \beta^{(3)}(1) = -6\sigma_4 = -\frac{21}{4} s_4. \tag{25}$$

From (14) and (7) it follows that

$$\int_0^1 \frac{\log^3 t}{1-t} dt = -\Psi^{(3)}(1) = -6 s_4. \tag{26}$$

Also from (4) we obtain, by an integration by parts,

$$\int_0^1 \frac{\log(1+t)\log^2 t}{1+t}\,dt = -A_4 + \frac{5}{16}s_4. \tag{27}$$

To obtain the 4th integral on the right of (24) we first modify (15). The integral (15) cannot be written as the difference of the two integrals

$$\int_0^1 \frac{\log 2}{1-t}\,t^{x-1}dt, \qquad \int_0^1 \frac{\log(1+t)}{1-t}\,t^{x-1}dt$$

since they do not separately exist. However, if we differentiate each side of (15) with respect to x, we may write

$$\eta^{(1)}(x) = \log 2 \int_0^1 \frac{\log t}{1-t}\,t^{x-1}dt - \int_0^1 \frac{\log(1+t)\log t}{1-t}\,t^{x-1}dt, \; R(x) > 0,$$

which by (14) becomes

$$\eta^{(1)}(x) = -\sigma_1\Psi^{(1)}(x) - \int_0^1 \frac{\log(1+t)\log t}{1-t}\,t^{x-1}dt. \tag{28}$$

This, with the help of (16), gives

$$\int_0^1 \frac{\log(1+t)\log t}{1-t}\,t^{x-1}dt \\ = -\sigma_1\Psi^{(1)}(x) - (1/2)\Psi^{(2)}(x) + \beta(x)\beta^{(1)}(x).$$

Hence finally,

$$\begin{aligned} &\int_0^1 \frac{(\log 1+t)\log^2 t}{1-t}\,dt \\ &= -\sigma_1\Psi^{(2)}(1) - (1/2)\,\Psi^{(3)}(1) + (\beta^{(1)}(1))^2 + \beta(1)\beta^{(2)}(1) \\ &= 2\sigma_1 s_3 - 3s_4 + \sigma_2^2 + 2\sigma_1\sigma_3 = 7\sigma_1 s_3/2 - 19s_4/8. \end{aligned} \tag{29}$$

From the 5th integral on the right of (24) we obtain, by an integration by parts, the relation

$$\int_0^1 \frac{\log^2(1+t)\log t}{1+t}\,dt = -\frac{1}{3}\int_0^1 \frac{\log^3(1+t)}{t}\,dt.$$

The value of the last integral is given by formula (7) of the table referred to above.* Thus,

* Rutledge and Douglass, *Table of definite integrals*, loc. cit.

$$\int_0^1 \frac{\log^2(1+t)\log t}{1+t}\,dt = -\frac{1}{2}A_4 + \frac{1}{32}s_4. \tag{30}$$

Hence, substituting the results of (25), (26), (27), (29) and (30) in (24) we have the relation

$$\begin{aligned}\int_0^{\pi/2} \phi \log^2 \cos \phi d\phi = \frac{83}{64}s_4 + \frac{7}{4}\sigma_1 s_3 - \frac{1}{4}A_4 \\ - \frac{1}{2}\int_0^1 \frac{\log^2(1+t)\log t}{1-t}\,dt.\end{aligned} \tag{31}$$

Now, from (6) by writing $\log(2\cos\phi) = \log 2 + \log\cos\phi$ we have, after a few reductions, the relation

$$\begin{aligned}\int_0^{\pi/2} \phi \log^2 \cos \phi d\phi = -2\sigma_1\int_0^{\pi/2} \cdot \phi \log \cos \phi d\phi \\ - \frac{3}{4}\sigma_1^2 s_2 + \frac{93}{64}s_4 + \frac{1}{4}A_4.\end{aligned} \tag{32}$$

The integral on the right of (32) may be evaluated completely by Serret's formula by differentiating the two sides of (17) once with respect to p, setting $p=0$, and then evaluating the resulting integrals in a manner analogous to that of the last paragraph. In this way it is found that

$$\int_0^{\pi/2} \phi \log \cos \phi d\phi = -\frac{7}{16}s_3 - \frac{3}{4}\sigma_1 s_2. \tag{33}$$

Then by (33) it is found that (32) becomes

$$\int_0^{\pi/2} \phi \log^2 \cos \phi d\phi = \frac{93}{64}s_4 + \frac{7}{8}\sigma_1 s_3 + \frac{3}{4}\sigma_1^2 s_2 + \frac{1}{4}A_4. \tag{34}$$

If now the right sides of (31) and (34) be equated, we get

$$\int_0^1 \frac{\log^2(1+t)\log t}{1-t}\,dt = -\frac{5}{16}s_4 + \frac{7}{4}\sigma_1 s_3 - \frac{3}{2}\sigma_1^2 s_2 - A_4. \tag{35}$$

This result, together with some of those above, serves to show that Serret's formula can be made a useful tool in studying certain types of definite integrals. Various other results may be obtained, both in connection with integrals of the orders treated here and of higher orders. However, we leave such considerations for treatment on another occasion.

The nature of the constant A_4 here remains undetermined just as in the papers of Rutledge and Douglass. Whether or not it can be rationally expressed in terms of the constants s_1, σ_1, s_3 and π is an open question. Some light may be thrown on the problem by a further study of the function $\xi_1(x)$ treated briefly by Nielsen.* His definition is as follows,

$$\xi_1(x) = \int_0^1 \frac{\log(1+t)}{1+t} t^{x-1} dt, \qquad R(x) > 0. \tag{36}$$

From this equation and (27) it follows that

$$A_4 = 5s_4/16 - \xi_1^{(2)}(1). \tag{37}$$

This in itself, of course, sheds no light but if a relation analogous to (16) could be found involving the function $\xi_1(x)$, it would seem that the question could be answered.

LEHIGH UNIVERSITY

THE COMPUTATION OF THE SMALLER COEFFICIENTS OF $J(\tau)$

HERBERT S. ZUCKERMAN†

The purpose of this note is to call attention to the fact that the first twenty-five coefficients $a_0, a_1, \cdots, a_{24}$ in the expansion

$$1728J(\tau) = e^{-2\pi i\tau} + \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau} \tag{1}$$

can be computed with relative ease, making use of H. Gupta's tables‡ of the partition function which extend to $n=600$.

From the multiplicator equation§ of fifth order of $J(\tau)$ we have

$$\begin{aligned} 1728J(\tau) = {} & y^{-1} + 6\cdot 5^3 + 63\cdot 5^5 y + 52\cdot 5^8 y^2 + 63\cdot 5^{10} y^3 \\ & + 6\cdot 5^{13} y^4 + 5^{15} y^5, \end{aligned} \tag{2}$$

with

* N. Nielsen, loc. cit., p. 233.

† Harrison Research Fellow.

‡ *A table of partitions*, Proceedings of the London Mathematical Society, vol. 39 (1935), pp. 142–149; *A table of partitions* II, Proceedings of the London Mathematical Society, vol. 42 (1937), pp. 546–549.

§ Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, vol. 2, p. 61, formula (11), with the values given in vol. 2, p. 64, (5) and vol. 1, p. 154, (1).