The Classical Groups. By Hermann Weyl. Princeton, University Press, 1939. 11+302 pp.

It is a curious fact that while almost all the textbooks on higher algebra written prior to 1930 devote considerable space to the subject of invariants, the recent ones written from the axiomatic point of view disregard it completely. Because of this neglect the phrase "invariant theory" is apt to suggest a subject that was once of great interest but one that has little bearing on modern algebraic developments. For this reason it is an important and original accomplishment that Professor Weyl has made here in connecting the theory of invariants with the main stream of algebra and in indicating that the subject has a future as well as a distinguished past.

In his treatment the theory of invariants becomes a part of the theory of representations. The most natural way to begin the study of invariants of a particular group is therefore to determine its representations. A large part of the book is concerned with this problem as it applies to the "classical" groups $GL(n)$, the full linear group; $O(n)$, the orthogonal, and $S(n)$, the symplectic (complex or abelian) group.

In discussing the representations of an abstract group $\mathfrak{g}$ one finds it convenient to adjoin to the set of representing matrices their linear combinations. The resulting set is an algebra, the enveloping algebra of the original set, and defines a representation of a certain abstract algebra, the group algebra, that is completely determined by $\mathfrak{g}$ and the field of the coefficients. In this way the theory of algebra is applicable. The author has gone somewhat beyond his immediate needs in discussing this domain. Consequently his book may serve also as an excellent introduction to this theory.

The book begins with a brief review of the basic concepts of field, abstract group, vector space, linear transformation (matrix) and representation of a group. These are used to give the following definition of an invariant: Let $x, y, \cdots$ be variable vectors varying in different representation spaces of an abstract group $\mathfrak{g}$ and let $A(s), B(s), \cdots$ be the corresponding matrices. A polynomial $f(x, y, \cdots)$ in the coordinates of $x, y, \cdots$ is an invariant of $\mathfrak{g}$ if $f(A(s)x, B(s)y, \cdots) = f(x, y, \cdots)$ for all s in $\mathfrak{g}$. Relative invariants and covariants are defined along similar lines. The central problems to which this definition gives rise are: (1) Given a class of invariants, to determine, if possible, a basis, that is, a finite number $I_1, I_2, \cdots, I_r$ of these invariants such that every invariant in the class is expressible as a polynomial in the I's. (2) To determine a fundamental set of relations obtaining between the basic invariants. The solution of these problems for vector invariants[1] of the unimodular group $SL(n)$ and of the orthogonal group is the subject of Chapter II. There are two essential steps in the proofs, the formal part by which the theorems are reduced by certain identities (Capelli's) to the cases where the number of vector variables of each kind does not exceed $(n-1)$ and the numerical argument which disposes of the reduced cases. The proofs are less direct for $O(n)$ than for $SL(n)$. For here it is necessary to suppose first that the underlying field is real and then to extend the results to arbitrary fields of characteristic 0 by using Cayley's parametrization of the "general" orthogonal matrix of determinant 1 as $(E-S)(E+S)^{-1}$ where E is the identity and S is an arbitrary skew-symmetric matrix.

The third chapter introduces the theory of algebras. The treatment given is based on the concept of a completely reducible matrix algebra. It is shown that any representation of an algebra $\mathfrak{A}$ of this type is completely reducible and there is a reciprocity between $\mathfrak{A}$ and the algebra $\mathfrak{B}$ consisting of the matrices commutative with those of $\mathfrak{A}$: $\mathfrak{B}$ is completely reducible and $\mathfrak{A}$ in turn may be characterized as the set of matrices commutative with those of $\mathfrak{B}$. Together with Schur's lemma this result yields Wedderburn's fundamental theorems. The methods used here are well suited for handling problems of the type discussed in this book, where algebras occur in concrete form as sets of matrices. More abstract methods would be required if one wished to obtain these theorems in their most general form. In the second half of the chapter an independent treatment of group algebras is given. Their complete reducibility and reciprocity with the commutator algebras is derived in a very elementary way. As a consequence the results, particularly the latter, are given a much more explicit form than in the general situation.

After this machinery has been set up, it is possible to determine the (integral) representations of $GL(n)$. One must establish first the complete reducibility of the tensor representation $A \to A \times A \times \cdots \times A$ (Kronecker product of f factors) and to split this representation into its irreducible parts. This is accomplished by showing that the enveloping algebra $\mathfrak{A}_f$ is the commutator of a certain representation of σ_f, the group algebra of the symmetric group on f letters. Explicit formulas for the idempotent elements of σ_f (Young symmetrizers)

[1] That is, functions of an arbitrary number of covariant and contravariant vectors where by a covariant vector we mean one belonging to the original representation $x \to Ax$ and contravariant refers to vectors in $\xi \to (A')^{-1}\xi$, A' the transposed matrix. There is no distinction between covariant and contravariant vectors if A is orthogonal.

yield the irreducible subspaces of tensor space. These results may be extended to the representation obtained by stringing along the diagonal the tensor representations of ranks 1, 2, $\cdots$, f. The enveloping algebra is here denoted as $\mathfrak{A}^{(f)}$. Now, if $A \rightarrow T(A)$ is any representation of $GL(n)$ whose coordinates are polynomials of degree not greater than f in the coordinates of A, then the enveloping algebra of $T(A)$ represents $\mathfrak{A}^{(f)}$. The general theory shows that $T(A)$ is completely reducible into parts similar to some of those contained in the tensor representations.

To obtain these results for $O(n)$ the author follows a procedure due to Brauer (used first by Weyl for $S(n)$). The complete reducibility of the tensor representation is established for the case of real fields. Its commutator algebra is determined by using the first main theorem for orthogonal invariants. Finally one obtains the enveloping algebra of the tensor representation by the reciprocity theorem. The tool by which the reality restriction is removed is again Cayley's parametrization. The real significance of the parametrization is brought out in the following beautiful theorem: The ideal generated by the relations defining an orthogonal matrix of determinant 1 is prime and has the generic zero $(E-S)(E+S)^{-1}$. To obtain the irreducible parts of tensor space relative to $O(n)$ a process of contraction is used. By this means the problem is reduced to one of decomposing certain spaces relative to the symmetric group. These results are then extended to the proper orthogonal group. The discussion of the symplectic group is reserved for a separate chapter. It parallels that of $O(n)$ and enables the reader to get a bird's-eye view of the entire structure as it appears up to this point.

In cases such as those under discussion here, in which the representations decompose into absolutely irreducible parts, a knowledge of these parts, though not their explicit construction, may be obtained by expressing the character of the given representation as a sum of primitive (irreducible) characters. This is done in Chapter VII for $GL(n)$, $O(n)$ and $S(n)$. The method is that of Weyl's "unitarian trick." One replaces these groups by certain compact Lie groups (thus, in place of $GL(n)$ one uses the group of unitary matrices). The characters of the latter may be obtained by using an invariant integration defined on the group manifolds. Since any algebraic relation with rational coefficients in the elements of the compact group $\mathfrak{u}$ holds also for the original group $\mathfrak{g}$, the formulas obtained for $\mathfrak{u}$ are valid for $\mathfrak{g}$ after being expressed as polynomials in the coefficients of the matrices. It is shown incidentally that the representations, determined by the earlier constructions form a complete set of continuous irreducible representations for the compact groups. As an illustration of the use of characters, the author determines the irreducible components of the Kronecker product of irreducible representations.

In the first part of Chapter VIII the general definition of an invariant is specialized to the classical case. Here the representation space is determined by the coefficients of a homogeneous polynomial. The symbolic method as well as certain irrational methods are discussed as tools to obtain the first fundamental theorem. Hilbert's basis theorem for polynomial ideals settles the question of the finiteness of relations among the basic invariants. The symbolic method is then used to extend these results to general orthogonal and symplectic invariants. The ideas in the last half of the chapter cluster around the concept of a Lie group and its Lie algebra. The local properties of these groups are all expressible in terms of the algebras and if the group is simply connected one may readily pass to its properties in the large. On the other hand Lie algebras may be studied independently of the group theory and questions analogous to those regarding representations, invariants, and so on, may be raised. For compact Lie groups the powerful integration method finds another application here in a very simple derivation of the first fundamental theorem for general invariants. The last chapter resumes the discussion of algebras. A number of interesting results concerning automorphisms and direct products are obtained by matrix methods.

We have confined our description to the broad outlines of the subject. Space does not permit a detailed account of the interesting bypaths indicated by the author. We may mention, for example, a new formulation of Klein's Erlanger program, the close connection between representation theory and the theory of almost periodic functions and Fourier series, the topology of the classical groups. There are ample indications in the notes and bibliography to enable the reader to pursue these questions further.

In short, the book is heartily recommended to the present generation of algebraists as an introduction to a rich and rather neglected field and to those educated in the classical tradition for an insight into important recent algebraic ideas and their applicability to familiar problems.

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Aspects of the Calculus of Variations. Notes by J. W. Green after lectures by Hans Lewy. Berkeley, University of California Press, 1939. 6+96 pp.

As indicated in the introduction, the goal of the lecturer was to