

A THEOREM CONCERNING CLOSED AND COMPACT POINT SETS WHICH LIE IN CONNECTED DOMAINS¹

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The purpose of this paper is to show that the following theorem holds in any space which satisfies Axioms 0, 1, and 2 of R. L. Moore's *Foundations of Point Set Theory*.²

If g denotes a point set, \bar{g} will be used to denote the set g together with all its limit points. For each positive integer n , G_n will denote the collection G_n of Axiom 1.

THEOREM. *If M is a closed and compact subset of a connected domain D , then there exists a compact continuum containing M and lying in D .*

PROOF. For each point P of D , there exists a region g_P of G_1 containing P such that \bar{g}_P is a subset of D . By Axiom 2, there exists a connected domain d_P containing P which is a subset of g_P . Let U_1 denote the collection of all domains d_P for each point P of D . The point set M is closed and compact, and hence, by Theorem 22 of Chapter I, it is covered by a finite subcollection W_1 of U_1 . By Theorem 77 of Chapter I, for each pair of domains x and y of W_1 there exists a simple chain xy whose links are domains of U_1 and whose first and last links are x and y respectively. Let V_1 denote the collection of all domains v such that for some two domains x and y of W_1 , v is a link of the chain xy . The sum of all the domains of the finite collection V_1 is a connected domain D_1 . Similarly, there exists a finite collection V_2 of connected domains such that if v is any domain of V_2 , then \bar{v} is a subset of some region of G_2 and of some domain of V_1 , and such that the sum of the domains of V_2 is a connected domain D_2 . This process can be continued. Thus there exists an infinite sequence V_1, V_2, V_3, \dots such that, for each n , (1) V_{n+1} is a finite collection of connected domains such that if v is any one of them then \bar{v} is a subset of some region of G_{n+1} and of some domain of V_n and of D , and (2) the sum of all the domains of V_n is a connected domain D_n containing M . By Theorems 79 and 80 of Chapter I, the set of all points common to all the sets of the sequence D_1, D_2, D_3, \dots is a compact continuum, and it contains M and lies in D .

A modification of this argument proves this theorem for a space which satisfies Axioms 0 and 1 and is locally arcwise connected.

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² American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. All references are to this book.